This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

Digitized by Google

Digitized by Google

MATHEMATICAL

QUESTIONS AND SOLUTIONS

FROM "THE EDUCATIONAL TIMES."

WITH MANY

PAPERS AND SOLUTIONS

IN ADDITION TO THOSE

PUBLISHED IN "THE EDUCATIONAL TIMES."

EDITED BY

CONSTANCE I. MARKS, B.A.

NEW SERIES.

VOLUME III.

LONDON:

FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

1903.

Digitized by Google

288.86

HARVARD COLLEGE
JUN 1903
LIBRARY.

Harvard.
N.S. III.)

LIST OF CONTRIBUTORS.

ABBOTT, R. C., B.A.; Marlborough College.
ADAIR, E. W.; Guernsey.
ALEXANDER, J. J., M.A.; Tavistock.
ALLEN, Rev. A. J. C., M.A.; Cambridge.
ANDERSON, Prof., M.A.; Queen's Coll., Galway.
ANTHONY, EDWYN, M.A.; Westwood, Hereford.
ANTONIO, J.; Penang.
APPLETON, Miss E.; Liverpool.
ARCHIBALD, R. C., M.A., Ph.D.; Sackville, N.B., Canada.
ARMENANTE, Professor; Pesaro.
ARNOLD, I.; Institution, Belfast.
AYRTON, Mrs.; Hyde Park, W.
BALL, Sir ROBERT STAWELL, LL.D., F.R.S., Professor of Astronomy and Geometry, Camb.
BALL, W. W. ROUSE; Cambridge.
BARNIVILLE, JOHN J., B.A.; Jessore, Bengal.
BARRETT, E. P., B.A.; Stroud.
BARTON, W. J., M.A.; Highgate, London.
BATEMAN, H.; Cambridge.
BATES, G.N., B.A.; Ramsgate.
BATTAGLINI, Prof. GIUSEPPE; Univ. di Roma.
BEMAN, Prof. W. W.; Univ. Michigan.
BESANT, W. H., D.Sc., F.R.S.; Cambridge.
BEYENS, Prof. IGNACIO, Lt.-Col. du Génie; Cadiz.
BHATTACHARYYA, Prof. N.
BHUT, Professor ATH BIJAH, M.A.; Dacca.
BICKERDIKE, C.; Allerton, near Castleford.
BIDDLE, D. (Editor, Vols. LXVII.–LXXV.); Charlton Lodge, Kingston-on-Thames.
BIGGS, R., M.A., LL.D.; Portora, Enniskillen.
BILLUPS, H. B.; Sandown, I. of W.
BIRCH, Rev. J. G., M.A.; Bunralty, co. Clare.
BIRTWISTLE, G., B.A., B.Sc.; Pemb. Coll., Camb.
BLAIKIE, J., M.A.; Norwood.
BLATER, JOSEPH; Baden Baden.
BLYTHE, W. H., M.A.; St. Mark's Lodge, Newnham.
BOLAND, J.; Sand Hill, Gibraltar.
BOOTH, Prof. W.; Hoogly College, Bengal.
BORDAGE, Prof. EDMOND; Coll. de Nantua.
BOURNE, Rev. C. W., M.A.; King's Coll. Sch.
BOUTON, V. J., B.Sc.; Hampton Wick.
BOWSER, Prof. E. A.; New Jersey, U.S.A.
BOYS, Prof.; Royal College of Science.
BRILL, J., M.A.; Cambridge.
BROCARD, H., Chef de Bataillon; Bar-le-Duc.
BROOKS, Prof. E.; Millersville, Pennsylvania.
BROWN, A. CRUM, D.Sc.; Edinburgh.
BROWN, Prof.; Providence, U.S.A.
BRUCE, Rev. ROBERT, D.D.; Huddersfield.
BRUNYATE, W. E.; Trinity Coll., Cambridge.
BRYAN, G. H., F.R.S.; Bangor.
BRYANT, R., D.Sc., B.A.; London Institution.
BUCK, EDWARD, M.A.; Oakhill, Bath.
BURKE, J., M.A.; Trinity College, Dublin.
BURNSIDE, Professor, M.A.; Univ. of Dublin.
BURSTALL, H. J. W.; King's Coll., London.
CAMA, B. N., M.A.; St. John's Coll., Cambridge.
CAMPBELL, J. E., M.A.; Hertford Coll., Oxford.
CANHAM, D.; Norwich.
CAPEL, H. N., LL.B.; Bedford Square, London.
CARMODY, W. P., B.A.; Clonmel Gram. School.
CARR, G. S., M.A.; Chavey Down, Bracknell, Berks.
CAVALLIN, Prof., M.A.; University of Upsala.
CAVE, A. W., B.A.; Magdalen College, Oxford.
CAVE, F. E.; Girton Coll., Cambridge.
CHAKRAVARTI, Professor BYOMAKESA, M.A.; Calcutta.
CHANDRA BASÚ, Professor, M.A.; Calcutta.
CHAPMAN, Prof. R. W., M.A.; Univ. of Adelaide, S. Australia.
CHARTRES, R.; Carter Street, Manchester.
CHRISTIE, R. W. D.; Stanley Coll., Liverpool.
CLARK, A.; Aberystwyth.
CLARKE, Colonel A. R., C.B., F.R.S.; Redhill.
CLAYTON, Prof., M.A.; Parkville, Melbourne.
COCHEZ, Professor; Paris.
COHEN, ARTHUR, M.A., Q.C., M.P.; Holland Pk.
COLSON, C. G., M.A.; University of St. Andrews.
CONSTABLE, Rev. W. J., M.A.; Uppingham.
COONDOO, Prof. MAHENDRA NATH; Benares.
COONEY, W. S.; High School, Dublin.
COOPER, T. M. A., M.A.; Cheltenham.
COTTERILL, J. H., M.A.; St. Alban's Mansions, W.
CRAWFORD, G. E., M.A.; Clifton, Bristol.
CREMONA, Prof. LUIGI; Rome.
CROFTON, Professor M. W., F.R.S.; Bognor.
CROKE, J. O'BYRNE, M.A.; Dublin.
CULLEN, Rev. J.; Manresa House, Roehampton, S.W.
CULLEY, Prof., M.A.; St. David's Coll., Lampeter.
CUNNINGHAM, ALLAN, Lt.-Col., R.E.; Kensington
CURJEL, H. W., M.A.; Southport.
CURTIS, Prof., M.A., S.J.; Univ. Coll., Dublin.
CZUBER, Dr. EMANUEL; Prag, Bohemia.
DALLAS, R. J.; Pemberton Gardens, N.
DANIEL, VINCENT, B.Sc.; Cardiff.
DANIELS, A. E., B.A.; Nottingham.
DARBOUX, Professor; Paris.
DATA, Prof. PROMATHANATH, M.A.; Delhi.
DAVIES, Prof., Brown Univ., Providence, U.S.A.
DAVIS, Rev. B., M.A.; Principal of Jai Narayan College, Benares.
DAVIS, R. F., M.A.; Acton Green.
DAWSON, H. G., M.A.; Christ Coll., Cambridge.
DE LONGCHAMPS, Professor; Paris.
DE VRIES, Dr. JAN; Utrecht.
DE WACHTER, Professor, M.A.; Schaarbeck.
DEY, Prof. NARENDRA LAL, M.A.; Calcutta.
DIBB, J. H., B.Sc.; Hendon.
DICK, Hon. G. R.; Registrar Gen., Mauritius.
DICKSON, J. D. H., M.A.; Peterhouse, Camb.
DICKSON, W.; Edinburgh.
DOBBS, W. J., M.A.; Earl's Court.
D'OCAGNE, MAURICE; Paris.
DROZ-FARNY, Prof. A.; Collège de Porrentruy.
DRURY, H. D., M.A.; Marlborough College.
DUPAIN, J. C.; Professor au Lycée d'Angoulême.
DURELL, Rev. C. V., M.A.; Fulbourn Rectory, near Cambridge.
EASTON, BELLE, B.Sc.; Lockport, New York.
EDMONDSON, Prof. T. W.; New York Univ.
EDWARDES, DAVID, B.A.; Bangor.
ELLIOTT, Prof. E. B., F.R.S.; F. Mag. C., Oxon.; Oxford.
ELTON, E. F., M.A.; Wellington College.
EMMERICH, Prof., Ph.D.; Mülheim-am-Ruhr.
ESCOTT, Prof. E. B.; Michigan, U.S.A.
EVE, A. E., M.A.; Marlborough College.
EVERETT, Prof. J. D., D.C.L.
EWIN, CECIL; Drumcondra, Dublin.
FINCKEL, R. F.; N. Lewisburgh, Ontario, U.S.A.
FLOOD, P. W.; Drumcondra, Dublin.
FORSYTH, Prof., Sc.D.; Cambridge.
FORTEY, H., M.A.; Clifton, Bristol.
FOSTER, F. W., B.A.; Chelsea.
FOSTER, Prof. G. CAREY, F.R.S.
FOSTER, W. S.; Hoddesdon, Herts.
FRANKLIN, Prof. CHRISTINE LADD, M.A., B.Sc.
FREETH, Rev. Dr.; Fotherby, Louth, Lincs.
GALLIERS, Rev. T., M.A.; Kirstead Rect., Norwich
GASKIN, C. H.; Birkenhead.
GENESE, Prof., M.A.; Univ. Coll., Aberystwyth.
GEORGI, MAURICE; Paris.
GERRANS, H. T., M.A.; Fel. & Tut. Wor. Coll., Oxon.
GHOSH, SRISH CHANDRA, M.A.; Calcutta.
GHOSH, Prof. U. CHANDRA; Allahabad.
GIBSON, T. PAUL; St. Cloud, France.
GLAISHER, J. W. L., F.R.S.; Fel. Trin. Coll., Camb.
GOLDENBERG, Professor, M.A.; Moscow.
GOPALACHARI, Prof.; Presidency Coll., Madras.
GORDON, A., M.A.; Barnwood Ho., Gloucester.
GOULD, S. C.; New Hampshire, U.S.A.
GRACE, G. M., B.A., B.Sc.; Hammersmith.
GRACE, J. H.; Peterhouse, Cambridge.
GRAHAM, R. A., M.A.; Trinity College, Dublin.
GRAHAM, THOMAS; Kilkeel, Co. Down.
GREENHILL, A. G., M.A., F.R.S.; Artil. C., W'wich.
GREENSTREET, W. J., M.A.; MarlingSch., Stroud.
GREENWOOD, JAMES M.; Kansas City.
GRIFFITH, GERTRUDE, B.A.; Southampton.
GRIFFITHS, D. T., M.A.; Caermarthen.
GRIFFITHS, G. J., M.A.; Fellow Ch. Coll., Camb.
GRIFFITHS, J., M.A.; Fellow of Jesus Coll., Oxon.
GROSS, W., LL.D.; Bournemouth.
GROVE, W. B., B.A.; Perry Bar, Birmingham.
GUNNING, S., J.P.; Cookstown, Co. Tyrone.
HADAMARD, Professor, M.A.; Paris.
HAIGH, E., B.A., B.Sc.; King's Sch., Warwick.
HALL, Professor ASAPH, M.A.; Washington.
HAMILTON, J. P.; Penang.

Digitized by Google

HAMMOND, J., M.A.; 14 Granville Square, W.C.
HANUMANTA RAU, Professor, B.A.; Madras.
HARDY, G. H., B.A.; Trinity Coll., Cambridge.
HARKEMA, C.; University of St. Petersburg.
HARLEY, Rev. R., M.A., F.R.S.; Forest Hill.
HEAL, W. E., M.A.; Marion, Ind., U.S.A.
HENDERSON, R., B.A., Queen's Coll., Belfast.
HEPPEL, GEORGE, M.A.; Ealing.
HERVEY, F. R. J., M.A.; Worthing.
HILL, Rev. E., M.A.; Cockfield, Suffolk.
HILL, F. W.; City of London School.
HILL, Prof. M. J. M., M.A., D.Sc., F.R.S.; Univ. Coll. Lond.
HILLHOUSE, Dr. W.; Newhaven, Conn., U.S.A.
HILLS, G. F. S.; Trin. Coll., Camb.
HILLYER, C. E., M.A.; Cambridge.
HIME, Lt.-Col.; Alexandria.
HINTON, C. H., M.A.; Cheltenham College.
HIPPISLEY, R. L., Major, R.E.; Farnborough.
HOBSON, Dr.; Cambridge.
HOLT, J. R., M.A.; Dublin.
HOOKER, J. H., M.A.; 184 Evering Road, N.E.
HOPKINS, Rev. G. H., M.A.; Stratton, Cornwall.
HOROBIN, C. J., B.A.; M.L.S.C.
HOWSE, G. F.; Balliol College, Oxford.
HUDSON, C. T., LL.D.; Manilla Hall, Clifton.
HUDSON, J. F.; Gunnersbury House, Acton.
HUDSON, W. H. H., M.A.; Prof. in King's Coll., Lond.
HUGHES, EDITH K. A.; Stroud.
HUGHES, W. R; Jesus College, Cambridge.
HUNTINGTON, E.V., Ph.D.; Camb., Mass., U.S.A.
IKIN, A. E., B.Sc.; Kettering.
ISSERLIS, L.; Christ's Coll., Cambridge.
JACKSON, F. H., M.A.; Hull.
JACKSON, Miss F. H.; Towson, Baltimore.
JACKSON, W. H., B.A.; Clare Coll., Camb.
JACOBS, H. J.; London Institution, Finsbury.
JAGO, G.; Plymouth Public School.
JENKINS, MORGAN, M.A.; Tunbridge Wells.
JOHNSON, Prof., M.A.; Annapolis, Maryland.
JOHNSTONE, W. J., M.A.; Univ. Coll., Aberystwyth.
JOLLIFFE, A. E., M.A.; St. David's College.
KAHN, A., B.A.; Mildmay Grove, N.
KALIPADU BASU, Prof., M.A.; Dacca College.
KATHERN, R. P., M.A.; Fleetwood, Bath.
KELVIN, Lord, F.R.S.; Largs, Ayrshire.
KENNEDY, D., M.A.; Catholic Univ., Dublin.
KITCHIN, F. L., B.A., R.N.; Exmouth.
KITCHIN, Rev. J. L., M.A.; Exmouth.
KITTUDGE, LIZZIE A.; Boston, United States.
KLEIN, Professor FELIX; Univ. Göttingen.
KNISELEY, ALEX.; Columbia City, Indiana.
KNOWLES, R., L.C.P.; Tottenham.
KOEHLER, J.; Rue St. Jacques, Paris.
KOLBE, Rev. Dr.; St. Mary's Coll., Cape Town.
KRISHNACHANDRA DE, Prof.; Pudukkottai.
KRISHNAMACHARRY, Prof., M.A.; Tiupati, India.
LACHLAN, R., M.A.; Newnham Terrace, Camb.
LAISANT, C. A.; Paris.
LAMB, R.; George Town, Penang.
LAMPE, Prof., Ed. of *Jahrb. der Math.*; Berlin.
LANGLEY, E. M., B.A.; Adelaide Sq., Bedford.
LASKER, E., Ph.D.; Manchester.
LAVERTY, W. H., M.A.; late Exam. in Univ. Oxford.
LAWRENCE, E. J.; Ex-Fell. Trin. Coll., Camb.
LAWSON, J. S., M.A.; Newcastle-on-Tyne.
LAYNG, A. E.; Grammar School, Stafford.
LEGABEKE, Professor; Delft.
LEIDHOLD, R., M.A.; Finsbury Park.
LEMOINE, E.; Avenue du Maine, Paris.
LEUDESDORF, C., M.A.; Fel. Pembroke C., Oxon.
LEWIS, JOHN; Cardiff.
LEWIS, WM., M.A.; Headmaster, Llandyssul.
LIAGRE, General; Rue Carotz, Brussels.
LONDON, Rev. H., M.A.; Wimbledon.
LONGCHAMPS, Professor de; Paris,
LONGHURST, Rev. E. S., B.A.; Newark.
LOWRY, W. H., M.A.; Blackrock, Dublin.
LOXTON, C. A.; Cannock.
LUDWIG, Professor; North Carolina, U.S.A.
MCALISTER, DONALD, M.A., D.Sc.; Cambridge.
MACAULAY, F. S., M.A.; St. Paul's School.
MCCLELLAND, W. J., B.A.; Prin. of Santry School.
MACCOLL, HUGH, B.A.; Boulogne.
MCCOY, Rev. R.; Stonyhurst.
MCCUBBIN, J., B.A.; Burgh Academy, Kilsyth.
MACDONALD, W. J., M.A.; Edinburgh.
MACFARLANE, Prof. A., D.Sc.; Univ. of Texas.
MCINTOSH, ALEX., B.A.; Bedford Row, London.
MACKAY, J. S.; Edinburgh.
MACLEOD, J., H.M.I.S., I.S.O.; Elgin.
MACMAHON, Major P. A., F.R.S.; R.M.Academy.
MCMAHON, Prof., M.A.; Cornell Univ., Ithaca.
MACMURCHY, A., B.A.; Univ. Coll., Toronto.
MCVICKER, C. E., M.A.; Kensington.
MADDISON, ISABEL, B.A.; Brynmawr, Pa., U.S.A.
MADHAVARAO, Prof., M.A.; Vizianagram.
MAINPRISE, B. W.; R. Naval School, Eltham.
MALET, J. C., M.A., F.R.S.; Dublin.
MANN, M. J. J.; Cromwell Road, London.
MANNHEIM, M.; Prof. à l'École Polytech. Paris.
MARKS, Miss CONSTANCE I., B.A. (Editor); W. Kensington.
MARTIN, ARTEMAS, M.A., Ph.D.; Washington.
MASSEY, W. H.; Twyford, Berks.
MATHEWS, Professor G. B., M.A.; Bognor.
MATZ, Prof., M.A.; King's Mountain, Carolina.
MELLOR, J. W., B.Sc.; Owens Coll., M'chester.
MERRIFIELD J., LL.D., F.R.A.S.; Plymouth.
MERRIMAN, MANSFIELD, M.A.; Yale College.
METCALFE, G., M.A.; Wimbledon.
MEYER, MARY S.; Girton College, Cambridge.
MILLER, W. J. CLARKE, B.A. (Editor, Vols. I.–LXVII.); Bournemouth.
MILNE, Rev. J. J., M.A.
MINCHIN, G. M., M.A.; Prof. in Coopers Hill Coll.
MITCHELL, HUGH, M.A.; Edinburgh.
MITCHESON, Rev. T., B.A.; Marazion.
MITTAG-LEFFLER, Professor; Stockholm.
MONCK, H. ST., M.A.; Trin. Coll., Dublin.
MOORE, H. K., B.A.; Trin. Coll., Dublin.
MOREL, Professor; Paris.
MORGAN, C., M.A.; R. Naval Coll., Greenwich.
MORLEY, Prof., Sc.D.; Johns Hopkins Univ., Baltimore.
MORRICE, G. G., M.A., M.D.; Weymouth.
MUGGERIDGE, G. D., B.A.; Pemb. Coll., Camb.
MUIR, THOMAS, M.A., F.R.S.; Cape Town.
MUIRHEAD, R. F.; Glasgow.
MUKHOPADHYAY, Prof. ASUTOSH, M.A., F.R.S.E.
MUKHOPADHYAY, Prof. SYAMADAS; Chinsurah.
MULCASTER, J. W., M.A.; Allendale Town.
MUNINDRA, NATH RAY, Prof. M.A., LL.B.
NANSON, Prof. E. J.; Melbourne.
NEALE, C. M.; Middle Temple, E.C.
NESBITT, A. M., M.A.; Melbourne.
NESBITT, H. A.; Hampstead.
NEUBERG, Professor; University of Liège.
NEWCOMB, Prof. SIMON, M.A.; Washington.
NISNER, I.; Westbourne Grove, W.
NIXON, R. C. J., M.A.; Royal Acad. Inst., Belfast.
NORTH, FRANCIS C.; Ashton-on-Mersey.
O'CONNELL, Major-General P.; Cheltenham.
OLDHAM, C. H., B.A., B.L.; Dublin.
OPENSHAW, Rev. T. W., M.A.; Clifton, Bristol.
ORCHARD, JOHN, M.A.
ORCHARD, Professor, M.A., B.Sc.; Hampstead.
O'REGAN, JOHN; New Street, Limerick.
ORPEUR, HERBERT; East Dulwich.
OWEN, J. A., B.Sc.; Tennyson St., Liverpool.
PANCHAPAGESA AIYAR, K.G.; Madras.
PANTON, A. W., M.A.; Fell. of Trin. Coll., Dublin.
PARANJPYE, R. P., B.A.; Poona, India.
PAROLING, Professor; Cornell University.
PEACHELL, F. H., B.A.; High Wycombe.
PEELE, T., A.I.A.; Manchester.
PEIRCE, Prof. B. O.; Harvard College.
PEIRIS, M. R.; Moratuwa, Ceylon.
PERRIN, EMILY, B.Sc.; St. John's Wood Park.
PHILLIPS, F. B. W.; Balliol College, Oxford.
PILLAI, Professeur, M.A.; Trichy., Madras.
PLAMENEWSKI, H., M.A.; Dahgestan.
POCKLINGTON, H. C., M.A.; Yorks Coll., Leeds.
POLIGNAC, Prince CAMILLE DE; Paris.
POLLEXFEN, H., B.A.; Windermere College.
POOLE, A. W.; London.
POOLE, GERTRUDE, B.A.; Cheltenham.
PORTELA, Señor FRANCISCO DE V.; Fregenal de la Sierra, Spain.
PRESCOTT, J., B.A.; Manningham Lane, Bradf'd.
PRESSLAND, A. J., M.A.; Academy, Edinburgh.
PRESTON, G. W., B.A.; Wheatley, near Halifax.
PRICE, H., M.A.; Penarth.
PRUDDEN, FRANCES E.; Lockport, New York.
PUCKLE, G. HALE, M.A.; Windermere.
PURSER, FREDK., M.A.; Fell. Trin. Coll., Dublin.
PURSER, Prof., M.A.; Queen's Coll., Belfast.
PUTNAM, K. S., M.A.; Rome, New York.

Digitized by Google

PYDDOKE, H. W., M.A.; Oxford.
QUIN, A. G. B.; Streatham.
QUINT, Dr.; Gravenhäge, Netherlands.
RADHAKRISHNAN, Prof., M.A.; Pudukkottai.
RAMA AIYANGAR, Prof.; Trichinopoly, Madras.
RAMACHANDRA ROW, Prof., M.A.; Trichinopoly.
RAMSEY, A. S., M.A.; Fettes Coll., Edinburgh.
RANGACHARI, V. SRINIVASA, B.A.; Government College, Kumbakonum, India.
RAU, ALBERT G., B.S.; Bethlehem, Pa., U.S.A.
RAU, V. RAGUNATH; Torupati, India.
RAWSON, ROBERT; Havant, Hants.
REAY, LIONEL E., B.A.; West Hampstead.
REES, E. W.; Penarth, Cardiff.
REEVES, G. M., M.A.; Lee, Kent.
REYNOLDS, B., M.A.; Cambridge.
RICE, JAMES; Craigie Road, Belfast.
RICHARDS, DAVID, B.A.; Christ's Coll., Brecon.
RICHARDSON, Rev. G., M.A.; Winchester Coll.
ROACH, Rev. T., M.A.; Twyford, Hants.
ROBB, A. A.; St. John's Coll., Cambridge.
ROBERTS, A. D., B.A.; Jesus Coll., Oxford.
ROBERTS, R. A., M.A.; New York.
ROBERTS, W. R. W., M.A.; Fell. of Trin. Coll., Dub.
ROBSON, H. C., B.A.; Sidney Sussex Coll., Camb.
ROGEL, Professor; Mittweida, Germany.
ROGERS, L. J., M.A.; Yorkshire Coll., Leeds.
ROWTON, Rev. S. J., M.A., Mus. Doc.; Folkestone.
ROY, Professor KALIPRASANNA, M.A.; Agra.
RUGGERO, SIMONELLI; Università di Roma.
RUSSELL, ALEXANDER, B.A.; Oxford.
RUSSELL, B.; Pembroke Lodge, Richmond.
RUSSELL, J. W., M.A.; Merton College, Oxford.
RUSSELL, R., M.A.; Fell. of Trin. Coll., Dublin.
RUSSELL, T.; Caius College, Cambridge.
RUTTER, E.; Sunderland.
SADLER, JOHN; Birkenhead.
ST. CLAIR, J. C., M.A.; Ryder Street, London.
SALMON, Rev. GEORGE, D.D., F.R.S.; Provost of Trinity College, Dublin.
SALMON, W. H., B.A.; Hampstead.
SANDBERG, C. F., M.A.; Oxford.
SANDERSON, Rev. C. M., M.A.; Brington Rectory.
SANJÂNA, Prof., M.A.; Bhavnagar, Bombay.
SARADARANJAN RAY, Prof., M.A.; Dacca.
SARKAR, Prof. BENI MADHAV, M.A.; Agra.
SARKAR, Prof. NILKANTHA, M.A.; Calcutta.
SAVAGE, Prof. T.; St. Patrick's Coll., Armagh.
SCHARAR, C. H.; Scranton, U.S.A.
SCHEFFER, Professor; Mercersbury Coll., Pa.
SCHOUTE, Prof. P. H.; University of Groningen.
SCOTT, A. W., M.A.; St. David's Coll., Lampeter.
SCOTT, Professor CHARLOTTE A., D.Sc.
SCOTT, GEORGE, M.A.; Greystones.
SCOTT, R. F., M.A.; Fell. St. John's Coll., Camb.
SEGAR, HUGH W.; Trinity Coll., Cambridge.
SEN, Prof. RAJ. MOHAN; RajshabyeColl., Bengal.
SEWELL, Rev. H., B.A.; Bury Grammar School.
SEYMOUR, W. R., M.A.; St. Leonards.
SHARPE, J. W., M.A.; Bournemouth.
SHEPHERD, Rev. A. J. P., B.A.; Fellow of Queen's College, Oxford.
SHIELDS, Prof., M.A.; Coopwood, Mississippi.
SIMMONS, Rev. T. C., M.A.; Grainthorpe Vicarage, North Thoresby S. O., Lincs.
SIRCOM, Prof. SEBASTIAN, M.A.; Stonyhurst.
SITARAMAIYAR, Prof., Hindu Coll., Tinnevelly.
SIVERLEY, WALTER; Oil City, Pennsylvania.
SKRIMSHIRE, Rev. E., M.A.; Llandaff.
SMITH, C., M.A.; Sidney Sussex College, Camb.
SMITH, Professor D. E.; Brockport, N.Y.
SMITH, J. G.; Portora, Enniskillen.
SMYLY, J. G.; Merrion Square, Dublin.
SOPER, H. E., B.A.; Highgate.
SPARK, Rev. F. M.; Lewisham, London.
STANHAM, W. C., B.A.; Kensington.
STEEDE, B.H., M.A.; Trinity College, Dublin.
STEGGALL, Prof. J. E. A., M.A.; Dundee.
STEPHEN, ST. JOHN, B.A.; Caius Coll., Cambridge.
STEWART, H., M.A.; Islington.
STOOPS, J. M., B.A.; Victoria College, Belfast.
STOOPS, W.; Newry.
STORMONTH, JAMES; Edinburgh.
STORR, G. G., M.A.; Clerk of the Medical Council.
STOTT, WALTER, B.A.; Royal Ins. Co., Liverpl.
STRACHAN, J. C.; Anerley, London, S.E.
STUART, T., D.Sc.; Belfast.
SWAMINATHA AIYAR, Professor, M.A.; Madras.
SYMONS, E. W., M.A.; Grove House, Walsall.
TANNER, Prof. H. W. L., M.A.; S. Wales Univ. Coll.
TARLETON, F. A., M.A.; Fell. Trin. Coll., Dublin.
TATE, JAMES, M.A.; Kedleston Road, Derby.
TAYLOR, Rev. CHARLES, D.D.; Master of St. John's College, Cambridge.
TAYLOR, F. G., M.A., B.Sc.; Nottingham.
TAYLOR, H. M., M.A., F.R.S.; Fellow Trin. Coll., Cambridge.
TAYLOR, J. H., M.A.; Little Trinity, Cambridge.
TAYLOR, W. W., M.A.; Oxford.
TERRY, Rev. T. R., M.A.; Ilsley Rectory.
THIRD, J. A., M.A.; Beith, Ayrshire.
THOMAS, A. E., M.A.; Merton College, Oxford.
THOMAS, Rev. D., M.A.; Garsington Rect., Oxford.
THOMPSON, C. H.; St. David's Coll., Lampeter.
THOMSON, Rev. F. D., M.A.; Ex-Fellow of St. John's Coll., Cambridge.
THOMSON, W. L.; Edinburgh.
THYAGARAGAIYAR, V. R., M.A.; Bangalore.
TIRELLI, Dr. FRANCESCO; Univ. di Roma.
TODD, E. WALTER; Belfast.
TRACHTENBERG, H. L.; Trin. Coll., Camb.
TRACHTENBERG, M. I., St. John's Coll., Camb.
TRAILL, ANTHONY, M.A., M.D.; Fell. T.C.D.
TUCKER, R., M.A.; Worthing.
VAN DER HEYDEN, A. F., M.A.; Middlesbrough.
VENKATA, Prof. K. RAMANAIYA; Madras.
VIGARIÈ, ÉMILE; Laissac, Aveyron.
VINCENZO, JACOBINI; Università di Roma.
VOSE, Professor G. B.; Washington.
WALENN, W. H.; Mem. Phys. Society, London.
WALKER, G. F.; Queens' College, Cambridge.
WALMSLEY, J., B.A.; Eccles, Manchester.
WARBURTON-WHITE, R., B.A.; Salisbury.
WARD, BEATRICE A., B.Sc.; Cheltenham.
WARD, F. L., B.A.; Hartlepool.
WARREN, A. T., M.A.; Usk Terrace, Brecon.
WARREN, R., M.A.; Trinity College, Dublin.
WASSON, J. C.; Kells, Meath.
WATHERSTON, Rev. A. L., M.A.; Bowdon.
WATSON, D., M.A.; Folkestone.
WATSON, Rev. H. W.; Ex-Fell. TrinityColl., Camb.
WATTS, J. O.; Middlesbrough.
WEBB, G. H.; Halifax.
WEBB, H. A.; Trinity College, Cambridge.
WEBSTER, H. W., M.D.; Chelsea.
WELLACOT, Rev. W. T., M.A.; Newton Abbot.
WERTSCH, FRANZ; Weimar.
WEST, J. WOODCOTE; New Cross.
WHALLEY, L. J., B.Sc.; Leytonstone.
WHAPHAM, R. H. W., M.A.; Manchester.
WHITE, E. A., M.A.; Eckington, Pershore.
WHITEHEAD, R. F.; Cambridge.
WHITESIDE, G. M.A.; Eccleston, Lancashire.
WHITWORTH, Rev. Preb. W. A., M.A.; London.
WIGGINS, Rev. T., B.A.; St. Beuno's Coll., St. Asaph,
WILCOX, Rev. A. M., M.A.; Branhope, Leeds.
WILKINSON, J. F.; Bacup.
WILLIAMS, A. E., M.A.; Texas.
WILLIAMS, D. J., M.A.; Worcester Coll., Oxon.
WILLIAMSON, B., M.A.; F. & T. Trin. Coll., Dub.
WILMOT, Rev. W. T., M.A.; Newton Abbot.
WILSON, Rev. J., M.A.; Bannockburn.
WILSON, Ven. Archdeacon J. M., M.A., F.G.S.
WILSON, Rev. J. R., M.A.; Royston, Cambs.
WOLSTENHOLME, Professor, M.A., Sc.D.
WOLTERS, J. B.; Groningen.
WOODALL, H. J., A.R.C.S.; Stockport.
WOODCOCK, T., B.A.; Twickenham.
WORKMAN, W. P., M.A.; Bath.
WORTHINGTON, W. R., M.A.; Southport.
WRIGHT, W. J.; New Jersey, U.S.A.
YOUNG, JOHN, B.A.; Portadown, Ireland.
YOUNGMAN, C. E., M.A.; Corinth House, Cheltenham.
ZERR, Professor, M.A.; Staunton, Virginia.

*** Of this Mathematical publication there have now been issued the seventy-five Volumes of the Original Series and three Volumes of the New Series. Each Volume contains, in addition to the papers and solutions that have appeared in the *Educational Times*, an equal quantity of new matter, consisting of further solutions, and of completions of, or additions to, solutions that have been partly published in the pages of the journal.

These volumes contain contributions, in all branches of Mathematics, from most of the leading Mathematicians in this and other countries.

New Subscribers may have any of these Volumes at Subscription prices.

Digitized by Google

CONTENTS.

Mathematical Papers, &c.

Page

Note on Question 14975. (W. H. Blythe, M.A.) 29

Addition to Mr. Blythe's Note. (G. N. Bates) 29

How to reduce $\int \frac{dx}{\sin x + \sec x}$ and $\int \frac{dx}{\cos x + \operatorname{cosec} x}$ to known Integrable Forms. (G. Scott, M.A.) 34

Geometrical Note. (R. F. Davis, M.A.) 57

A Corollary to Euclid III. 22. (Rev. C. Taylor, D.D.) 73

Note on two connected Problems in Mean Value. (R. Chartres)... 78

Problems whose Arithmetical Solutions are indicated by a Knowledge of Algebra. (W. R. Worthington, M.A.) 89

Geometrical Solution of Question 14961. (R. F. Davis, M.A.)... 101

The Integration of $\int \frac{d\theta}{\sec\theta + \operatorname{cosec}\theta}$. (G. Scott, M.A.) 107

Taking Euclid III. 36 as the Definition of a circle, show that the ordinary Definition is a Theorem. (J. Macleod, H.M.I.S.) 107

Solved Questions.

2039. (Professor Wolstenholme.) — Find the mean value of the perimeters of inscribed triangles in a given circle 114

3619. (Omega.)—Given the sides of three squares inscribed in a triangle, determine the triangle 79

5473. (H. W. Harris.)—Prove that, if

$$F(m, n, p) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha^m & \beta^m & \gamma^m & \delta^m \\ \alpha^n & \beta^n & \gamma^n & \delta^n \\ \alpha^p & \beta^p & \gamma^p & \delta^p \end{vmatrix},$$

then $F(m, n, p)\{F(1, 2, 3)\}^2 = \begin{vmatrix} F(m, 2, 3), & F(1, m, 3), & F(1, 2, m) \\ F(n, 2, 3), & F(1, n, 3), & F(1, 2, n) \\ F(p, 2, 3), & F(1, p, 3), & F(1, 2, p) \end{vmatrix}$,

and generalize for a similar determinant of any order 82

Digitized by Google

5702. (Rev. Prebendary W. A. Whitworth, M.A.)—A and B play at a game which cannot be drawn, and the odds are $\mu : \nu$ in favour of A winning any assigned game. They agree to play until either A shall have won by scoring m games ahead of B, or B shall have won by scoring n games ahead of A. Show that, when A has scored p games more than B, the odds in favour of his winning the match are

$$\mu^{m+n} - \mu^{m-p}\nu^{n+p} : \mu^{m-p}\nu^{n+p} - \nu^{m+n} \quad \ldots\ldots\ldots\ldots\ 75$$

5734. (A. Martin.)—A has m dollars and B n dollars, and they play at a game for which it is $b : a$ that B shall win any one game; prove that the chances for the ultimate success of each player are

$a^n(b^m - a^m)/(b^{m+n} - a^{m+n})$ and $b^m(b^n - a^n)/(b^{m+n} - a^{m+n})$ respectively 75

5778. (Wilkins.)—If a hyperboloid of one sheet pass through three given lines in space, obtain its centre by a geometrical construction... 120

7296. (T. Muir, M.A., F.R.S.E.) — Show that, if Ω denote the bordered Hessian

$$\begin{vmatrix} 0 & a & b & c \\ a & \mathrm{P} & \mathrm{Q} & \mathrm{S} \\ b & \mathrm{Q} & \mathrm{R} & \mathrm{T} \\ c & \mathrm{S} & \mathrm{T} & \mathrm{U} \end{vmatrix},$$

(1) the equation $\left(\frac{d}{d\mathrm{T}}x - \frac{d\Omega}{d\mathrm{S}}y\right) = 4c^2\Omega$ denotes two straight lines passing through the two points in which the conic

$$\mathrm{P}x^2 + 2\mathrm{Q}xy + \mathrm{R}y^2 + 2\mathrm{S}x + 2\mathrm{T}y + \mathrm{V} = 0$$

is cut by the straight line $ax + by + c = 0$; and (2) the equation of the straight line which completes the inscribed quadrilateral of which the said three lines are sides is

$$\begin{vmatrix} 0 & a & b & c \\ a & \mathrm{P} & \mathrm{Q} & \mathrm{S} \\ b & \mathrm{Q} & \mathrm{R} & \mathrm{T} \\ c & \mathrm{S} & \mathrm{T} & 2c\mathrm{S}/a \end{vmatrix} ax + \begin{vmatrix} 0 & a & b & c \\ a & \mathrm{P} & \mathrm{Q} & \mathrm{S} \\ b & \mathrm{Q} & \mathrm{R} & \mathrm{T} \\ c & \mathrm{S} & \mathrm{T} & 2c\mathrm{T}/b \end{vmatrix} by - \begin{vmatrix} 0 & a & b & c \\ a & \mathrm{P} & \mathrm{Q} & \mathrm{S} \\ b & \mathrm{Q} & \mathrm{R} & \mathrm{T} \\ c & \mathrm{S} & \mathrm{T} & 0 \end{vmatrix} c = 0$$

..... 117

7390. (W. S. McCay, M.A.)—If the straight lines joining the feet of perpendiculars on the sides of a triangle ABC from any point on the circumscribed circle cut the sides at angles A′, B′, C′, show that (1) its equation in areal coordinates referred to the triangle is

$$x \cot \mathrm{A}' + y \cot \mathrm{B}' + z \cot \mathrm{C}' = 0;$$

(2) this equation is a cubic in a single parameter (either cot A′, cot B′, or cot C′), and the envelope of the line is therefore a curve of the fourth degree with three cusps; (3) the equation of the line and its first and second differentials are all satisfied when $\mathrm{A}' + \mathrm{B}' + \mathrm{C}' = (2n+1)\frac{1}{2}\pi$, and therefore the tangents at the cusps are inclined to each other at angles of 120° [the angles being measured so that $\mathrm{B}' - \mathrm{C}' = \mathrm{A} + \pi$, $\mathrm{C}' - \mathrm{A}' = \mathrm{B} - \pi$, $\mathrm{A}' - \mathrm{B}' = \mathrm{C} - \pi$]..... 70

8091. (A. Gordon, M.A.)—If ABCD is a quadrilateral, M, N the mid-points of AC, BD, prove that (1) MN cuts AB (in P), DC (in R) in segments inversely proportional; (2) if MN cuts CB, AD in Q, S, then

$$\mathrm{MP} : \mathrm{MR} = \mathrm{MQ} : \mathrm{MS} \quad \text{and} \quad \mathrm{NP} : \mathrm{NR} = \mathrm{NS} : \mathrm{NQ} \quad \ldots\ldots\ 68$$

8276. (Professor Catalan.)—Démontrer la formule

$$\tfrac{1}{4}\pi = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \ldots \mp \frac{1}{2n-1} - 2n\int_0^{\frac{1}{2}\pi} \cos 2nx \log_e(2\cos\tfrac{1}{2}x)\,dx \quad \ldots\ 65$$

Digitized by Google

8871. (Professor Mahendra Nath Ray, M.A., LL.B.)—If

$$S_r = \frac{1}{2r+1} - \frac{1}{2n+2r+1} + \frac{1}{4n+2r+1} - \frac{1}{6n+2r+1} + \dots \text{ ad inf.},$$

n being a positive integer, show that

$$S_0 - kS_1 + \frac{k(k-1)}{2!}S_2 - \frac{k(k-1)(k-2)}{3!}S_3 + \dots \text{ to } k+1 \text{ terms}$$

$$= \int_0^{\pi} \frac{x \sin^{2k+1} x\, dx}{(1+\cos^{2n} x)} \quad \dots\dots 116$$

9076. (Professor Nash, M.A.)—If $A\omega$, $B\omega$, $C\omega$, $A\omega'$, $B\omega'$, $C\omega'$ meet the circum-circle in X, Y, Z, X′, Y′, Z′, show that (1) the triangles XYZ, X′Y′Z′ are similar and equal to ABC, and are Tucker's triangles of another triangle A′B′C′, similar and similarly situated to ABC, the centre of similitude being the pole of $\omega\omega'$ with respect to the Brocard circle; and (2) area A′B′C′ : area ABC = $\sin^2 3\omega : \sin^2 \omega$ 115

9290. (A. E. Jolliffe.)—Find the sum to n terms and also to $a+1$ terms of

$$1 - a^x + \frac{a^x(a^x-1^x)}{1^x \cdot 2^x} - \frac{a^x(a^x-1^x)(a^x-2^x)}{1^x \cdot 2^x \cdot 3^x} + \dots,$$

and deduce the value of

$$1 - \frac{a^x - r^x}{(r+1)^x} + \frac{(a^x-r^x)\{[a^x-(r+1)^x]\}}{(r+1)^x(r+2)^x} - \dots$$

to $(a-r+1)$ terms, where a is not less than r, a, r, x being integers 55

9309. (Professor Hudson, M.A.)—A cage, mass M, is drawn up a mine by an engine, at first with uniform acceleration, then with uniform velocity v, and then for a distance equal to the first portion with uniform retardation; the whole time of ascent is t, and the greatest horse-power at which the engine is worked is H. Prove that the depth of the mine is

$$\frac{550Hgvt - Mv^2(gt+v)}{(550H - Mv)g} \quad \dots\dots 43$$

9773. (B. Peirce.)—A un certain jeu, le joueur A. gagne un point chaque fois qu'il tire une boule blanche d'une urne qui contient 3 boules blanches et 2 boules noires; chaque fois qu'il tire une boule noire, il perd tous ces points, et B., qui ne fait jamais de tirage, en gagne un. Quelle est la probabilité pour A. de faire h points le premier? Cas particulier où $h = 3$ 89

9781. If $ax+by+cz+dw=0$, and $a'x+b'y+c'z+d'w=0$, where a, b, c, d, a', b', c', d' are functions of a parameter θ, be the equations to a line, this line will generate a ruled surface of order $m+n$, where m and n are the orders of the two given equations as functions of θ. Especially examine the case when $m=n=1$, and show that a second set of lines exists in this case 34

9822. (Professor G. B. M. Zerr.)—If $y = x^{n-1}\log x$, prove that, when r is not $> n$,

$$\frac{d^r y}{dx^r} = (n-1)(n-2)\dots(n-r)\,x^{n-r-1}\left\{\log x + \left(\frac{1}{n-1} + \frac{1}{n-2} + \dots + \frac{1}{n-r}\right)\right\};$$

and hence show that

$$(1+x)^{n-1}\log(1+x) = x + \frac{(n-1)(n-2)}{1.2}\left\{\frac{1}{n-1}+\frac{1}{n-2}\right\}x^2$$
$$+ \frac{(n-1)(n-2)(n-3)}{1.2.3}\left\{\frac{1}{n-1}+\frac{1}{n-2}+\frac{1}{n-3}\right\}x^3 + \dots$$
$$+ \left\{\frac{1}{n-1}+\frac{1}{n-2}+\frac{1}{n-3}+\dots+\frac{1}{1}\right\}x^{n-1}$$
$$+ \frac{x^n}{u}\left\{1 - \frac{1.x}{n+1} + \frac{1.2.x^2}{(n+1)(n+2)} - \frac{1.2.3.x^3}{(n+1)(n+2)(n+3)} + \dots\right\}$$

..... 82

Digitized by Google

9873. (J. O'Byrne Croke, M.A.)—If the sections of an ellipsoid through the axis of x be projected orthogonally into circles on planes passing through one of the extremities of that axis, those circles lie on the same sphere, and the envelope of their planes is the cone

$$\frac{y^2}{b^2(a^2-c^2)} + \frac{z^2}{c^2(a^2-b^2)} = \frac{(a-x)^2}{b^2c^2} \quad \ldots\ldots 35$$

9946. (Professor Genese, M.A.)—Through a point within a parallelogram four circles are described containing the sides as chords. If any two of these circles through the same vertex be equal, all four are ... 93

10157. (R. W. D. Christie.)—If p be a prime number >2, prove

$$\{(2p!/2 \times p! \times p!) - 1\} = p^2(\mathrm{M}),$$

where M is the sum of $\frac{1}{2}(p-1)$ integral square numbers 33

10198. (John Griffiths, M.A.)—The coordinates x, y, z being proportional to the perpendicular distances from the sides of a triangle ABC, prove that (1) the locus of a point (x, y, z) such that its pedal circle (*i.e.*, the circle which passes through the feet of the perpendiculars drawn from the point in question upon the sides of the triangle) touches the nine-point circle is the cubic

$$x(y^2-z^2)\cos A + y(z^2-x^2)\cos B + z(x^2-y^2)\cos C = 0;$$

and (2) trace this curve 29

10236. (Artemas Martin, LL.D.)—What is the greatest number of equal spheres, each 1 inch in diameter, that can be packed in a hollow sphere whose internal diameter is 3 inches? 61

10342. (Professor Nash, M.A.)—A variable circle passes through a fixed point A on a conic, and meets the conic again in B, C, D; the Simson-line of A with respect to the triangle BCD passes through a point whose position is independent of the position and magnitude of the circle. This point lies on the diameter through the image of A with respect to the axis; in the rectangular hyperbola it coincides with the centre... 119

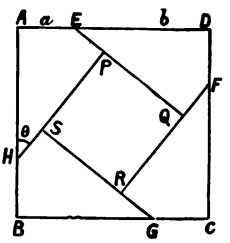
10933. (W. J. C. Miller, B.A.)—If P, Q be two random points inside a circle whose centre is C, find the average of (1) the perimeter, (2) the area, (3) the sum of the squares on the sides of the triangle CPQ 114

12369. (Professor Sylvester.)—Find the polar equation (*a*) to the general involute of the logarithmic spiral, (*b*) to the *second* involute of the circle..... 54

13967. (R. Tucker, M.A.)—In the figure, ABCD is a given square,

$$AE = DF = CG = BH = a,$$

and PQRS is another square. Find when PQRS is a *maximum* (θ variable).



..... 65

14144. (D. Biddle.)—The sum to infinity of the reciprocals of triangular numbers being 2, find the value of a constant, x, to be added to each triangular, such that the sum of the reciprocals shall be $4\pi - 1/x$ 98

Digitized by Google

14671. (R. Knowles.)—If A_r is the coefficient of x^r in the expansion of $(1-x)^{-n}$, prove that the sum of the series $1+A_1^2x+A_2^2x^2+A_3^2x^3+\ldots$ ad inf. is a function of the form

$$(1+p_1^2x+p_2^2x^2+\ldots+p_{n-2}^2x^{n-2}+x^{n-1})/(1-x)^{2n-1},$$

in which p_r is the coefficient of x^r in the expansion of $(1+x)^{n-1}$, *e.g.*, if $n=5$, the sum of the series

$$1+5^2x+15^2x^2+35^2x^3+\&c. = (1+16x+36x^2+16x^3+x^4)/(1-x)^9 \ldots 56$$

14780. (Rev. J. Cullen.) — A given line cuts the circumcircle in P and Q. Prove that (1) the asymptotes of the isogonal transformation of PQ are at right angles to the Simson lines of P and Q. (2) If PQ passes through the circumcentre, then the asymptotes are the Simson lines. (3) If PQ be $\Sigma a\lambda=0$ and (p, q, r) the trilinear co-ordinates of P (or Q), then the asymptotes are of the form $\Sigma a\lambda p^2=0$ 26

14812. (R. Knowles.)—If

$$F_1x/(1-x)^n = 1+2^{n-1}x+3^{n-1}x^2+\ \ldots,$$

and

$$F_2x/(1-x)^n = 1+3^{n-1}x+5^{n-1}x^2+\ \ldots,$$

show that $F_1x.F_2x$ is a symmetrical function of the degree $(2n-3)$ in x, the sum of whose coefficients $= \{(n-1)(n-2)\ldots 2\}^2.2^{n-1}$, *e.g.*, if $n=4$,

$$F_1x.F_2x = (1+27x+116x^2+116x^3+27x^4+x^5)/(1-x)^8 \ldots 71$$

14830. (R. W. D. Christie.)—Let r be the radius of the base of a segment of a sphere or a paraboloid, and h the height; P, S the solidity of the paraboloid and sphere respectively. Prove $P(3r^2+h^2)=3r^2S$, and, if $h=r$, then $4P=3S$ 104

14875. (Lt.-Col. Allan Cunningham, R.E.)—Show that

$$N_1 = x^8+mx^4y^4+y^8$$

is resolvable into factors when $m=(\frac{1}{4}\mu^4-2\mu^2+2)$; also that

$$N_2 = x^8+x^6y^2+mx^4y^4+x^2y^6+y^8$$

is resolvable into factors when $m=\frac{1}{4}(\mu^2-3)^2$.

Ex.— $N_1 = 100002501$, $N_2 = 101090101$ 87

14877. (Robert W. D. Christie.)—Factorize

$$3e^4+\{f(e-f)\}^2.$$

E.g.: $1483181587 = 24841 \times 59707$ 60

14881. (Corrected form.) (V. Daniel, B.Sc.)—Determine in Cartesian co-ordinates the family of plane curves related to two focal points P and Q, such that a particle projected through P shall, after rebounding from any point on one such curve, proceed in the line joining Q to that point. Coefficient of rebound $= e$; $PQ = 2a$. Taking for origin the middle point of PQ, show that the lines $x \pm y\sqrt{e} = a$ are common tangents at the focus Q 96

14883. (The late Professor Wolstenholme, M.A., Sc.D.) — Prove that

$$\int_0^\infty \frac{\log x}{1-x^n}dx = -\left(\frac{\pi}{n\sin\pi/n}\right)^2, \quad \int_0^\infty \frac{\log x}{1+x^n}dx = -\cos\frac{\pi}{n}\left(\frac{\pi}{n\sin\pi/n}\right)^2,$$

$$\int_0^\infty \frac{(\log x)^3}{1-x^n}dx = -2\left(\frac{\pi}{n\sin\pi/n}\right)^4\{1+2\cos^2\pi/n\},$$

$$\int_0^\infty \frac{(\log x)^3}{1+x^n}dx = -\left(\frac{\pi}{n\sin\pi/n}\right)^4(5\cos\pi/n+\cos^3\pi/n),$$

n being greater than 1; and generally that

$$\int_0^\infty \frac{(\log x)^{2r+1}}{1-x^n}dx = -\theta^{2r+2}\left(\frac{d}{d\theta}\right)^{2r}\left(\frac{1}{\sin^2\theta}\right),$$

and

$$\int_0^\infty \frac{(\log x)^{2r+1}}{1+x^n}dx = -\theta^{2r+2}\left(\frac{d}{d\theta}\right)^{2r+1}\left(\frac{1}{\sin\theta}\right),$$

where $\theta=\pi/n$ 84

Digitized by Google

14885. (Professor Sanjána, M.A.)—Let K_1 be the pole of the side BC of a triangle ABC with regard to the circum-circle, and O the circum-centre; let D, E, F be the mid-points of BC, CA, AB respectively; and K_1A_1, K_1B_1, K_1C_1 perpendiculars to OD, OE, OF respectively. Then the six points $K_1 \equiv A_1, B_1, C_1$, B, C, O lie on a circle. So also the similar points $K_2 \equiv B_2, C_2, A_2$, C, A, O and $K_3 \equiv C_3, A_3, B_3$, A, B, O lie on circles. [These circles have been called the ex-Brocardal circles of the triangle.] The triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ are each inversely similar to ABC, and in perspective with it; they are also in perspective two by two, and their axes of perspective meet in a point. [These triangles have been called ex-Brocardal triangles.] The ex-Brocardal triangles form a system of similar figures whose circle of similitude is the circum-circle ABC; whose invariable points are A′, B′, C′ on the circum-circle diametrically opposite to A, B, C, and whose invariable circles are the ex-Brocardal circles (so that every three collinear corresponding points of the figures lie severally on these circles) 105

14915. (Lt.-Col. Allan Cunningham, R.E.)—Show that

$$N = m^2\left\{(1-\nu)xy\right\}^4 + \nu\left\{x^2 - m(1+\nu)y^2\right\}^4$$

is factorisable. *Ex.*: $\frac{1}{2}N = 1 + \frac{1}{2}(xy)^4$ is factorisable when $x^2 - 3y^2 = 1$ 30

14917. (A. M. Nesbitt, M.A.)—Three conics S_1, S_2, S_3 have a common point O. The remaining intersections of S_2S_3 are the vertices of a triangle Δ_1, those of S_3S_1 of a triangle Δ_2, and those of S_1S_2 of a triangle Δ_3. Prove that the nine sides of these three triangles all touch the same conic U. Or, reciprocally: Given three parabolas, their common tangents, taking the curves in pairs, will form three triangles (one for each pair) whose nine vertices lie on a conic 86

14918. (Rev. T. Roach, M.A.)—If from S′, the second focus, a line be drawn parallel to SP, meeting the ellipse in P′ and SY in Z, show that SP′Z is an isosceles triangle 111

14930. (Émile Lemoine.)—Trouver dans un triangle ABC un point M tel que si l'on mène par M une parallèle à BC coupant AB en A_b, AC en A_c; une parallèle à CA coupant BC en B_c, BA en B_a; une parallèle à AB coupant CA en C_a, CB en C_b, les deux triangles $A_bB_cC_a$, $A_cB_aC_b$ soient égaux 76

14955. (Aletrop.)—Donner la règle pour trouver tous les *ternes* de nombres consécutifs chacun desquels soit une somme de deux carrés entiers; *e.g.*: $72 = 6^2 + 6^2$, $73 = 8^2 + 3^2$, $74 = 7^2 + 5^2$ 41

14961. (H. Bateman.)—The axis of a parabola circumscribing a triangle is a Simson line with respect to the triangle formed by joining the middle points of the sides 62

14982. (Professor Umes Chandra Ghosh.)—Prove that

$$\int_0^\infty \frac{\sin\left(\frac{1}{4}\pi + bx\right)}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{b}} \quad \ldots\ldots 54$$

14987. (T. Muir, M.A., F.R.S.)—Given

$$u \equiv (a, b, c, d\,\mathcal{X}\,x, y)^3 + e = 0,$$

show that $$\left(\frac{d^2y}{dx^2}\right)\left(\frac{du}{dx}\right)^3 = 2e\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ y^2, & -xy, & x^2 \end{vmatrix},$$

and generalize 44

14988. (The late Professor Cayley, F.R.S.)—If

$$l^2 + m^2 + n^2 + \ldots, \quad l'^2 + m'^2 + n'^2 + \ldots$$

are each = 0, and s, s' denote positive integers, show that the integral

$$\int (lx + my + nz + \ldots)^s (l'x + m'y + n'z + \ldots)^{s'}\,dx\,dy\,dz\ldots,$$

Digitized by Google

over the interior of the hyper-sphere $x^2+y^2+z^2+\ldots = 1$ is = 0 or = a numerical multiple of $(ll'+mm'+nn'+\ldots)^s$, according as s' is not, or is, $= s$ 94

14992. (Professor Neuberg.) — Par un point fixe A on mène une sécante quelconque qui coupe une courbe donnée Δ en B et une seconde courbe donnée Δ′ en B′. Les tangentes menées en B à Δ et en B′ à Δ′ se rencontrent en un point C. Construire la tangente en C de la courbe engendrée par ce point lorsque la sécante tourne autour de A ... 38, 53

14999. (R. Chartres.)—If the in-circle of a triangle passes through the centroid, (1) show that $5\Sigma(a^2) = 6\Sigma(ab)$ (Question 12641); (2) give an easy method of writing down in integers the sides of any number of such triangles; and (3) find the mean value of the in-radius as the smallest side takes all possible values between 1 and 2 units 44

15001. (A. M. Nesbitt, M.A.)—Two conics U and V intersect in ABCO, and through O two arbitrary straight lines OP, OQ are drawn; OP cuts U in P and V in Q′; OQ cuts U in P′ and V in Q. Prove that the four straight lines OP, OQ, PP′, QQ′ and the three sides of the triangle ABC all touch one and the same conic 86

15002. (R. Knowles, B.A.)—From a point T (h, k), tangents TP, TQ are drawn to meet the parabola $y^2 = 4ax$ in P and Q. Prove that (1) the coordinates of the symmedian point of the triangle TPQ are

$$[(a+h)(k^2+2ah),\ k(k^2+2ah+6a^2)]/2(k^2-ah+3a^2);$$

(2) if PQ be a focal chord, this point is on the tangent at the vertex... 51

15011. (Lionel E. Reay, B.A.)—Given the circumcircle and orthocentre, construct the triangle, being given also (1) a point on one of the sides, (2) one angle of the triangle..... 67

15012. (Professor Sanjána, M.A.) — Let P be any point on the circumcircle of ABC; draw a chord QR of the circle perpendicular to the Simson line of P; then PR shall be perpendicular to the Simson line of Q, and also PQ to that of R. Prove also that these three lines concur at the point midway between the orthocentres of the triangles PQR and ABC..... 26

15013. (A. M. Nesbitt, M.A.)—If there be three parabolas P, Q, R, and if the common tangents of Q and R form a triangle A, those of R and P a triangle B, and those of P and Q a triangle C, prove that the nine vertices of A, B, C are on a conic (which may be called the "nine-point conic" of the three parabolas). Show also that the four "nine-point conics" of four parabolas (taking them three at a time) pass through a common point. This theorem is of course also true, if for parabolas we substitute "conics having a common tangent," but the enunciation becomes of necessity more verbose 86

15017. (Professor Cochez.) — On donne une circonférence O et un point fixe A. On joint A à un point B de la circonférence, et sur AB on construit un triangle ABM semblable à un triangle donné. Lieu de M quand B décrit la courbe 50

15021. (Aletrop.) — Ranger les nombres 1, 2, 3, ..., 8 de manière que la différence entre deux quelconques d'eux ne soit pas égale en valeur absolue à la différence de leurs ranges 30

15031. (Aletrop.)—Résoudre en nombres entiers $x^{-2}+y^{-2} = z^{-2}$ 106

Digitized by Google

15032. (R. Knowles). — In De Morgan's *Calculus*, p. 257, the following examples are given :—

$$2.3.4+3.4.5+4.5.6=\frac{4.5.6.7-1.2.3.4}{4}=204;$$

$$2.3+3.4+4.5+5.6=\frac{5.6.7-1.2.3}{3.1}=68.$$

Show that these results can be obtained by means of the coefficients in the expansion of $(1-x)^{-n}$ 25

15033. (R. Tucker, M.A.)—Three circles are given by

$$x^2+y^2+C=0,\quad x^2+y^2+Ax=0,\quad x^2+y^2+By=0.$$

Find the area of the triangle formed by the polars of their radical centre with respect to the three circles 81

15038. (Professor N. Bhattacharyya.) — If Ω, Ω' be the Brocard points of a triangle ABC, and O be the centre of the Brocard ellipse, prove that

$$\begin{aligned}&(\Omega A^2-\Omega' A^2)/c^2+(\Omega B^2-\Omega' B^2)/a^2+(\Omega C^2-\Omega' C^2)/b^2\\&=(\Omega' A^2-\Omega A^2)/b^2+(\Omega' B^2-\Omega B^2)/c^2+(\Omega' C^2-\Omega C^2)/a^2\\&=2\,(1/b^2+1/c^2)(\Omega A^2-OA^2)+2\,(1/c^2+1/a^2)(\Omega B^2-OB^2)\\&\qquad+2\,(1/a^2+1/b^2)(\Omega C^2-OC^2)\\&=1-16\Delta^2/(b^2c^2+c^2a^2+a^2b^2),\end{aligned}$$

where Δ is the area of the triangle..... 66

15043. (Professor Cochez.)—Étudier les courbes

(1) $x^4-x^3y+4x^2-xy+1=0$, (2) $x^4y^2-x^3+4x^2y^2-x+y^2=0$... 40

15047. (Rev. Prebendary W. A. Whitworth, M.A.)—Prove that the lowest value of N for which the equation $x^2+3y^2=N$ has 24 solutions in positive integers, zero excluded, is N = 214396 28

15051. (R. Knowles.)—Prove that the sum to r terms of the series $1-6+15-28\ldots\pm r(2r-1)$, when r is an even number, $=-\frac{1}{2}r(2r+1)$, and, when r is an odd number, $=\frac{1}{2}(r+1)(2r-1)$ 63

15054. (G. F. S. Hills.)—There are six confocals to $x^2/a^2+y^2/b^2=1$, such that quadrilaterals circumscribed to this ellipse are inscribed in the confocal, and the six values of λ $[x^2/(a^2+\lambda)+y^2/(b^2+\lambda)=$ given 1] are by $(\lambda^2-a^2b^2)(\lambda^2+2b^2\lambda+a^2b^2)(\lambda^2+2a^2\lambda+a^2b^2)=0$ 59

15055. (Rev. T. Roach, M.A.)—If two ellipses have equal major axes and a common vertex, show that they meet in only one other point 51

15056. (Professor Neuberg.)—Trouver le lieu du sommet d'un angle circonscrit à une parabole, les côtés de cet angle étant parallèles à deux diamètres conjugués quelconques d'une conique donnée 36

15058. (Professor Nanson.)—If OP, OQ are small equal arcs of a curve and its circle of curvature at O, show that the difference of the chords of these arcs is $\frac{1}{24}\kappa\kappa' OP^4$, where κ is the curvature and $\kappa'=d\kappa/ds$ 91

15060. (Professor Morley.)—Prove that the circumcircle and nine-point circle of a triangle are inverse as to the director circle of the maximum inscribed ellipse 38

15062. (Professor Sanjána, M.A.)—On the circumcircle of a triangle ABC a fixed point P is taken, and QR is a chord of the circle parallel to the Simson line of P : prove that PR is parallel to the Simson line of Q, and PQ to that of R. The triangle formed by the three Simson lines is homothetic to the triangle PQR : find the locus of the homothetic centre as QR moves parallel to itself 26

Digitized by Google

15068. (R. W. D. Christie.) — Solve generally the following Christmas puzzle:—Find four integers a, b, c, d, such that

(1) $ad \equiv bc \equiv 1 \mod p$, (2) $a^2+b \equiv c+d^2 \equiv 0 \mod p$,

(3) $a^2c+1 \equiv bd^2+1 \equiv 0 \mod p$, (4) $bc^2+d^2 \equiv 0 \mod p$,

where p = prime of form $4m-1$, and give *all* the solutions 92

15069. (Aletrop.)—N et k étant des entiers donnés, de combien de manières différentes a-t-on en nombres entiers $N = a^2+kb^2$, lorsque cela soit possible? 36

15072. (R. Tucker, M.A.) — PSQ, RST are orthogonal focal chords of a parabola. O, O′ are the respective co-normal points (*i.e.*, PO, QO are normals, and also RO′, TO′). Show that the join of O, O′ passes through a fixed point, that O, O′ lie on the same parabola, and find the locus of the intersection of OO′ and the join of the third co-normal points of (P, Q) and (R, T) 39

15073. (A. M. Nesbitt, M.A.)—If we call the conic which touches the nine sides of the triangles Δ_1, Δ_2, Δ_3 the "nine-tangent conic" of the three original conics, prove that, if *four* conics P, Q, R, S be such that all pass through one point, while not more than two pass through any other point, then the four "nine-tangent conics" of QRS, RSP, SPQ, and PQR have a common tangent. The following particular case of the reciprocal of Question 14917 ought to admit of a solution by purely geometrical methods. Two parabolas P, P′ have a common focus S, and a circle C touches their common tangent. If the common tangents to P and C and to P′ and C form triangles Δ and Δ', prove that the point S and the six vertices of Δ and Δ' all lie on a circle 87

15080. (Professor Nanson.)—Lines drawn through the vertices of a triangle Δ divide the angles into the segments α, α'; β, β'; γ, γ', and form a triangle Δ'. If R, R′ are the radii of the circum-circles of Δ, Δ', prove that $$\frac{R'}{R} = \frac{\sin\alpha \sin\beta \sin\gamma - \sin\alpha' \sin\beta' \sin\gamma'}{\sin(\beta+\gamma)\sin(\gamma+\alpha')\sin(\alpha+\beta')},$$ and deduce the theorems of Ceva and Menelaus 74

15084. (A. M. Nesbitt, M.A.) — If u_r denote the chance that, of r letters intended for r different recipients and placed *at random* into r envelopes correctly addressed, not one shall reach its proper destination (disregarding errors of delivery, &c.), it is fairly obvious *a priori* that

$$u_n + u_{n-1}/1! + u_{n-2}/2! + \ldots + u_0/n! = 1.$$

Solve this difference equation *directly*; *i.e.*, prove that u_n = the first $n+1$ terms of the expansion of e^{-1} without reference to the problem from which it originally sprang 64

15086. (Lt.-Col. Allan Cunningham, R.E.) — Factorize completely (into prime factors) $N = (60{,}000^6+1)$ 114

15087. (Professor E. B. Escott.) — In *Nature* for May 6, 1897, we have the following "Sieve for Primes" given:—"In order that a number n be prime, it is necessary and sufficient that the $\frac{1}{2}(n+1)$-th term of the following recurrent series give a remainder 1 when divided by n:—

1, 4, 11, 29, 76, 199, 521, ...,

where the scale of relation is $u^n = 3u_{n-1} - u_{n-2}$." Show, by a numerical example, that this condition is not sufficient 46

15088. (Rev. T. Wiggins, B.A.)—(1) Of numbers between 1 and 20, how many sets of 3, 4, or 5 can be found such that the square of one number in each set equals the sum of the squares of the remaining numbers in the same set? Also, if the numbers from 1 to 100 are taken,

Digitized by Google

how many sets of 3 numbers satisfy the above conditions? No set should be a multiple of any other. (2) Find in *two* ways $a_1, a_2, \ldots a_9$, such that $a_1^2 = a_2^2 + a_3^2 = a_4^2 + a_5^2 = a_6^2 + a_7^2 = a_8^2 + a_9^2$, where *each* of the a's lies between 1 and 100, but the suffixes have no reference to order of magnitude 57

15092. (R. Tucker, M.A.)—ABC and A'B'C' are a triangle and its medial triangle. Ω, Ω'; Ω_1, Ω_2 are the Brocard points; and K, K' the symmedian points. Prove that $\Omega\Omega_1$, $\Omega'\Omega_2$ the join of the Brocard ellipse centres, and KK' pass through the common centroid. Also $\Omega\Omega_2$, $\Omega'\Omega_1$ intersect in $a^2\alpha = b^2\beta = c^2\gamma$ 32

15094. (Professor Umes Chandra Ghosh.)—If tangents be drawn to Kiepert's hyperbola of a triangle at its vertices and orthocentre, prove that the three diagonals of the quadrilateral formed by these tangents intersect in pairs at the feet of the perpendiculars of the triangle drawn from its vertices to the opposite sides..... 101

15096. (Robert W. D. Christie.) — Find two distinct sets of triangles equal in area and infinite in number, the respective products of whose circumscribed and inscribed radii are as the longest and shortest sides of one set 37

15097. (H. Bateman.)—PP', QQ' are two pairs of isogonal conjugate points. PQ, P'Q' meet in R; PQ', P'Q meet in R'. Prove that R and R' are isogonal conjugate points..... 28

15103. (J. S. Lawson, M.A.)—Show that the sum of the harmonic series $\frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b} + \ldots$ to n terms $> \frac{2n}{2a+(n-1)b}$ 48

15104. (R. Knowles.)—Prove that the sum to r terms of the series

$$1 - 10 + 35 - 84 + \ldots \pm \tfrac{1}{2}r(4r^2-1)$$

is $-\frac{1}{8}r(4r^2+6r-1)$ when r is even and $\frac{1}{8}(4r^3+6r^2-r-3)$ when r is odd 75

15107. (I. Arnold.) — There are two opposite inclined planes each inclined to the horizon at an angle of 15°, viz., AB and BC; and touching each other at B. The planes are 100 feet long each; and a small perfectly smooth metallic ball rolls down AB and then ascends BC. In what time will the ball come to rest between the two planes? 58

15108. (R. W. D. Christie.)—Prove by two distinct methods the theorem $2^n \equiv 1 \bmod \frac{1}{2}(p-1)$, when p is a prime of form $4m-1$ and n the number of its primitive roots 103

15109. (Lt.-Col. Allan Cunningham, R.E.) — Find the least solution in *positive* integers of $x^2 + y^2 = mz^2$, where $m = (3\mu - 1)$, a positive integer > 2, and $x \neq y$; and x, y, z have no common factor ... 74, 109

15110. (Professor Sanjána, M.A.)—Prove that the solution in rational numbers (integral or fractional) of the equation $M^2 - 2xN^2 = x^2 - 1$ will serve to factorise quantities of the form $N_1^4 + N_2^4$; and show that x, supposed a positive integer, will always be of the form $4q^2 + 1$ 49

15114. (A. M. Nesbitt, M.A.) — If a, b, c be real and positive, prove that $a/(b+c) + b/(c+a) + c/(a+b) \geq \frac{3}{2}$ 37

15118. (Professor Neuberg.)—On mène à deux cercles donnés A, B des tangentes a, b telles que la bissectrice de l'angle ab soit parallèle à une droite donnée m. Trouver le lieu du point de rencontre C des droites a, b, et construire la tangente à ce lieu au point C..... 76

Digitized by Google

15119. (H. Bateman.)—ξ, η, ζ are the distances of a point from the angular points of a triangle. Prove that the two points for which $\xi^l\eta^m\zeta^n$ is a maximum are the foci of the ellipse whose areal equation is

$$\sqrt{(lx)} + \sqrt{(my)} + \sqrt{(nz)} = 0 \quad \ldots\ldots\ldots\ldots 110$$

15126. (A. M. Nesbitt, M.A.) — Prove that the sum of all the integers less than $(p^2+1)^n$ which are divisible by p and not by p^2 is $(p-1)/2p^2\left\{(p^2+1)^n-1\right\}^2$, p and n being integral, and that the sum of their squares is $\left[\left\{(p-1)\,X\right\}/6p^2\right](2X^2-p^2)$, where $X \equiv (p^2+1)^n-1$ 119

15128. (Robert W. D. Christie.)—Let a, b be any two consecutive members of the series (continuant) 1, 2, 3, 5, 8, 13, &c. Then

$$4a^3b-4ab^3+1 = a_n^2.$$

It is required to find the series a_n 71

15130. (Professor Langhorne Orchard, M.A., B.Sc.) — An equilateral triangle, of side-length $= a$, is thrown "at random," from a height $= na$, upon an indefinite horizontal plane which is ruled with equidistant horizontal lines, crossed with equidistant vertical lines, so as to form a number of squares, each of side-length $2a$. Find the "probability" that a side of the triangle will cut one of the lines ... 47

15131. (C. E. Youngman, M.A.)—Given a semicircle on diameter BC, draw an ordinate PN, and on NP produced take A such that $BA^2 = BC\,(BN+NP)$. Prove that the triangle ABC has

$$\tan^2 A = \tan B \tan C;$$

Brocard angle $90°-A$; Brocard axis parallel to BC, &c. 99

15136. (Professor Cochez.)—Etudier les courbes

(1) $(x-y)^2x+2(x-y)y+1=0$, (2) $y^2=(x^4-1)/x^2$ ... 80

15142. (Professor Nanson.)—If

$$x+y+z+w=0, \quad ax+by+cz+dw=0,$$
$$(a-d)^2(b-c)^2(xw+yz)+(b-d)^2(c-a)^2(yw+zx)$$
$$+(c-d)^2(a-b)^2(zw+xy)=0,$$

prove that $x:y:z:w = (d-b)(d-c)(b-c) : (d-c)(d-a)(c-a)$
$: (d-a)(d-b)(a-b) : (b-c)(c-a)(a-b)$
..... 118

15143. (J. S. Lawson, M.A.)—Sum the series

$$3-20x-30x^2-290x^3-\ldots \text{ ad inf.},$$

where $x<\frac{1}{3}$, the series being recurring 70

15144. (R. Knowles.)—Prove that the sums of a finite number (r) of terms in the two series

$$1+(m+1)x+(2m+1)x^2+\ldots+(mr-m+1)x^{r-1}$$

and $$1+(m+1)^2x+(2m+1)^2x^2+\ldots+(mr-m+1)^2x^{r-1},$$

m being any positive integer, are respectively

$$[1+(m-1)x-(mr+1)x^r+(mr-m+1)x^{r+1}]+(1-x)^2$$

and $[1+(m^2+2m-2)x+(m-1)^2x^2-(mr+1)^2x^r$
$+(2m^2r^2-2m^2r+4mr-m^2-2m+2)x^{r+1}-(mr-m+1)^2x^{r+2}]+(1-x)^3$
..... 52

15145. (R. Chartres.) — Three points are taken at random within a triangle and joined. Find the mean value of the nth power of the area of the triangle thus formed 103

Digitized by Google

15146. (Professor Nanson.) — If (A, B, C) is the adjugate of the determinant (a, b, c), prove that

$$(BC, CA, AB) = -(bc, ca, ab)(a, b, c)^2 \ldots\ldots 108$$

15150. (Professor Langhorne Orchard, M.A., B.Sc.) — Find the product of the two series

(1) $1+3+7+13+21+31+\ldots$ to n terms;

(2) $1+13+73+241+601+1261+\ldots$ to n terms 79

15158. (H. Bateman.) — O is a fixed point on a curve, P a variable point, G is the centre of gravity of the arc OP. Prove that the radius of curvature of the locus of G is $GP^2/s \sin\chi$, where s = arc OP and χ = angle between GP and the tangent at P 120

15169. (A. M. Nesbitt, M.A.) — A man tosses a coin; he scores 1 for each head that turns up and 2 for each tail. Find the chance of his scoring exactly n 102

15172. (Professor Nanson.)—Sum to infinity the series

$$\frac{1}{1.2}-\frac{1}{3.4}+\frac{1}{6.7}-\frac{1}{8.9}+\frac{1}{11.12}-\ldots \ldots\ldots 112$$

15182. (G. Heppel, M.A.) — The three forces represented by the perpendiculars from the symmedian point to the sides of a triangle are in equilibrium 108

15183. (A. F. van der Heyden, M.A.)—If a mass m be placed at the centre of a sphere S, the potential due to m, at any point external to S, will be equal to that due to a distribution over S equal in sum to m. Hence deduce that, if a surface S include any number of particles, and if N be the change in the normal force due to these particles on crossing the surface, a distribution over S of density $N/4\pi$ will produce the same potential throughout all space external to S as the masses referred to 90

15199. (Professor Neuberg.) — Trouver le lieu du centre d'une quadrique passant par quatre points fixes et dont les axes sont parallèles à des droites données 95

15216. (R. C. Archibald, M.A., Ph.D.) — The *trisectrix* of Maclaurin ($r\cos\frac{1}{3}\phi = C$ or $r\sin 2\theta + a\sin 3\theta = 0$), the *logocyclic curve* or *strophoide* ($r\cos\theta + a\cos 2\theta = 0$), and the *cubic* of Tschirnhausen ($r^{\frac{1}{3}}\cos\frac{1}{3}\theta = a^{\frac{1}{3}}$) are nodal cubics. Express their equations in the form $X^{\frac{1}{3}} + Y^{\frac{1}{3}} + Z^{\frac{1}{3}} = 0$, where X, Y, Z are linear functions of the Cartesian co-ordinates..... 113

Digitized by Google

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

15032. (R. KNOWLES.) — In DE MORGAN'S *Calculus*, p. 257, the following examples are given :—

$$2.3.4+3.4.5+4.5.6=\frac{4.5.6.7-1.2.3.4}{4}=204;$$

$$2.3+3.4+4.5+5.6=\frac{5.6.7-1.2.3}{3.1}=68.$$

Show that these results can be obtained by means of the coefficients in the expansion of $(1-x)^{-n}$.

Solutions (I.) *by* J. PRESCOTT, B.A.; (II.) *by* T. PEELE, A.I.A., *and* A. W. POOLE, B.A.

(I.) We have $(1-x)^{-n}=(1-x)(1-x)^{-(n+1)}$,

or $1+nx+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+\ldots$

$$=(1-x)\left\{1+(n+1)x+\frac{(n+1)(n+2)}{2!}x^2+\frac{(n+1)(n+2)(n+3)}{3!}x^3+\ldots\right\}.$$

Hence, equating the sum of the coefficients of the powers of x up to x^r,

$$n+\frac{n(n+1)}{2!}+\frac{n(n+1)(n+2)}{3!}+\ldots+\frac{n(n+1)\ldots(n+r-1)}{r!}=\frac{(n+1)(n+2)\ldots(n+r)}{r!}-1.$$

By putting $n=4$, $r=3$, we get the first result, and by putting $n=3$, $r=4$, we get the second.

(II.) Let $S=\sum_{\beta=0}^{\beta=r-1}(p+\beta)(p+\beta+1)(p+\beta+2)\ldots(p+\beta+m-1)$.

Now, since the coefficient of x^m in the expansion of $m!\,(1-x)^{-n}$ is

$$n(n+1)\ldots(n+m-1),$$

S = the coefficient of x^m in the expansion of $m!\sum_{n=p}^{n=p+r-1}(1-x)^{-n}$,

i.e., in $m!\,(1-x)^{-(p+r-1)}\frac{1-(1-x)^r}{1-(1-x)}$.

Digitized by Google

Hence S = the coefficient of x^{m+1} in $m!\cdot\left[(1-x)^{-(p+r-1)}-(1-x)^{-(p-1)}\right]$

$$= \frac{(p+r-1)(p+r)(p+r+1)\ldots(p+r+m-1)-(p-1)p(p+1)\ldots(p+m-1)}{m+1}.$$

Putting $p=2$, $m=3$, $r=3$,

$$S = 2.3.4+3.4.5+4.5.6 = \frac{4.5.6.7-1.2.3.4}{4} = 204.$$

Putting $p=2$, $m=2$, $r=4$,

$$S = 2.3+3.4+4.5+5.6 = \frac{5.6.7-1.2.3}{3} = 68.$$

14780, 15012, 15062. 14780.—(Rev. J. CULLEN.)—A given line cuts the circumcircle in P and Q. Prove that (1) the asymptotes of the isogonal transformation of PQ are at right angles to the SIMSON lines of P and Q. (2) If PQ passes through the circumcentre, then the asymptotes are the SIMSON lines. (3) If PQ be $\Sigma a\lambda = 0$ and (p, q, r) the trilinear coordinates of P (or Q), then the asymptotes are of the form $\Sigma a\lambda p^2 = 0$.

15012. (Professor SANJÁNA, M.A.)—Let P be any point on the circumcircle of ABC; draw a chord QR of the circle perpendicular to the SIMSON line of P; then PR shall be perpendicular to the SIMSON line of Q, and also PQ to that of R. Prove also that these three lines concur at the point midway between the orthocentres of the triangles PQR and ABC.

15062. (Professor SANJÁNA, M.A.)—On the circumcircle of a triangle ABC a fixed point P is taken, and QR is a chord of the circle parallel to the SIMSON line of P: prove that PR is parallel to the SIMSON line of Q, and PQ to that of R. The triangle formed by the three SIMSON lines is homothetic to the triangle PQR: find the locus of the homothetic centre as QR moves parallel to itself.

Solutions (I.) *by* C. E. YOUNGMAN, M.A.; (II., 15012) *by* R. F. DAVIS, M.A.; (III., 15012) *by* H. W. WEBSTER, M.D., *and* S. CONSTABLE.

(I.) For 15012, see Question 14613, Vol. LXXV., p. 107. In the figure for it, take p diametrically opposite to P; its pedal line will be perpendicular to that of P, and therefore parallel to QR; and pQ, pR are perpendicular to PQ, PR—*i.e.*, parallel to the pedals of R and Q; so that the triangle pQR exhibits the properties of Question 15062, first part. For the second, the pedals of p, Q, R bisect Hp, HQ, HR at L, M, N (H being the orthocentre of ABC), and they are parallel to the sides of LMN; therefore they form a triangle $p'Q'R'$ with sides double those of LMN, or equal to those of pQR. Hence the homothetic centre bisects pp'. And when QR moves parallel to itself p' describes the pedal of P; therefore the locus required is the straight line half way between this pedal and p.

Call P′, Q′ the points isogonal to P, Q; they are at infinity, so that the asymptotes of the hyperbola ABCP′Q′ (the isogonal transformation of PQ) are the tangents at P′, Q′, and are parallel to AP′, AQ′. Now, whatever the position of P, AP′ is perpendicular to one of the sides of the pedal triangle of P; hence the first part of 14780. In part (2) the

Digitized by Google

hyperbola is rectangular, passing through H; and its centre O is on the nine-point circle; therefore R, diametrically opposite to H, lies on the circumcircle. Hence the isogonal of R is at infinity and on PQ; therefore the pedal of R is perpendicular to PQ. Consequently (by 15012), since R is the orthocentre of PQR, the pedals of P and Q meet at O; and each, being perpendicular to one asymptote, coincides with the other. Thirdly, the line joining the isogonals of two points $(\alpha_1\beta_1\gamma_1)$ and $(\alpha_2\beta_2\gamma_2)$ on $\Sigma a\lambda = 0$ has evidently the equation $\Sigma a a_1 a_2 \lambda = 0$. Hence the equation of the tangent at P′ is $\Sigma a p^2 \lambda = 0$.

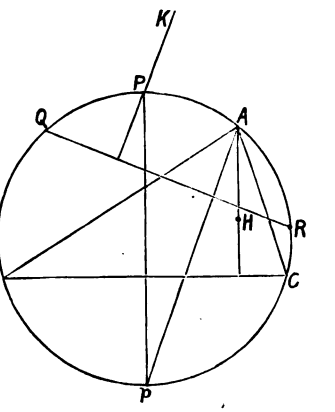
(II.) If Pp be the chord of the circumcircle perpendicular to BC, the Simson line of P is parallel to Ap, and bisects the join of P to the orthocentre H (Milne and Davis, p. 52). Then, since Pp, BC are perpendicular,

arc BP + arc Cp
= semi-circumference,
= arc AQ + arc Rp,

since Ap, QR are perpendicular; hence

arc BQ = arc AP + arc CR;

which it will be similarly found is also the condition for RP being perpendicular to the Simson line of Q, &c.



Let K be the orthocentre of PQR. The Simson line of P bisects PH and is perpendicular to QR, or parallel to PK. It therefore bisects HK.

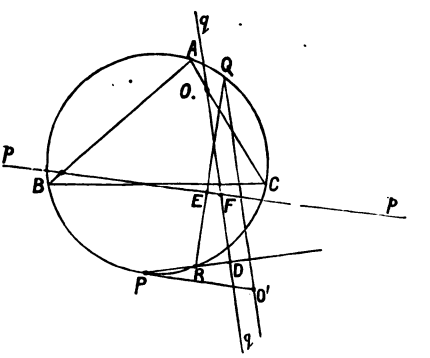
(III.) Let pp, the Simson line of P, meet qq, the Simson line of Q, in F; QR cut pp at right angles in E, and PR cut qq in D. By a well known theorem the Simson lines of P and Q must meet at the same angle subtended by the chord PQ on the circumcircle; therefore

∠ DFE = PRQ;

therefore FERD is a cyclic quadrilateral; therefore

∠ FDR = a right angle,

being supplementary to FER.



Let O′ be the orthocentre of PQR. Then QR is perpendicular to PO′. Therefore PO′ is parallel to pp. By a well known theorem the Simson

Digitized by Google

line of P bisects the line PO, if O be the orthocentre of ABC. Therefore pp bisects OO′. Similar results follow by using the SIMSON line of R in the first theorem, and the SIMSON lines of Q and R in the second.

15097. (H. BATEMAN.)—PP′, QQ′ are two pairs of isogonal conjugate points. PQ, P′Q′ meet in R; PQ′, P′Q meet in R′. Prove that R and R′ are isogonal conjugate points.

Solution by Professor SANJÂNA, M.A.

Let P, Q, R have the coordinates $(a_1b_1c_1)$, $(a_2b_2c_2)$, $(a_3b_3c_3)$ respectively; then their isogonal conjugates P′, Q′, R′ will be $(a_1^{-1}b_1^{-1}c_1^{-1})$, $(a_2^{-1}\ldots)$, $(a_3^{-1}\ldots)$ respectively. Also let PQR, P′Q′R be straight lines; then the determinants $(a_1b_2c_3)$ and $(a_1^{-1}b_2^{-1}c_3)$ are each zero. It can then be shown that the determinants $(a_2^{-1}b_3^{-1}c_1)$ and $(a_3^{-1}b_1^{-1}c_2)$ are each zero; hence the points Q′, R′, P are in a straight line, and so also are R′, P′, Q. From which the proposition follows.

15047. (Rev. Prebendary W. A. WHITWORTH, M.A.)—Prove that the lowest value of N for which the equation $x^2+3y^2 = N$ has 24 solutions in positive integers, zero excluded, is N = 214396. [The PROPOSER desires to withdraw Question 14998, as the statement made is true only in a restricted sense.—EDITOR.]

Solutions (I.) *by* R. W. D. CHRISTIE; (II.) *by* Lt.-Col. ALLAN CUNNINGHAM, R.E.

(I.) $N = 214396 = 2^2 . 7 . 13 . 19 . 31$. The primes 7, 13, 19, 31 are each of form $3M+1 = a^2+3b^2$. Now, by an extension of EULER's theorem, we have $(a^2+nb^2)(c^2+nd^2) = (ac \pm nbd)^2+n(ad \mp bc)^2$, and *four other* forms, making six in all, and $n = 3$; we easily deduce

$$7 \times 13 = 8^2+3.3^2 = 4^2+3.5^2 = (\tfrac{1}{2}7)^2+3.(\tfrac{5}{2})^2 = (\tfrac{1}{2}1)^2+3.(\tfrac{8}{2})^2$$
$$= (\tfrac{1}{2}8)^2+3.(\tfrac{1}{2})^2 = (\tfrac{1}{2})^2+3.(\tfrac{1}{2}1)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(A)},$$
$$19 \times 31 = 17^2+3.10^2 = 1^2+3.14^2 = (\tfrac{1}{2}7)^2+3.(\tfrac{7}{2})^2 = (\tfrac{1}{2}3)^2+3.(\tfrac{3}{2}7)^2$$
$$= (\tfrac{4}{2}1)^2+3.(\tfrac{1}{2}5)^2 = (\tfrac{4}{2}3)^2+3.(\tfrac{1}{2}3)^2 \quad \ldots\ldots\ldots\ldots\text{(B)}.$$

Therefore N = the following twenty-four solutions, where, for brevity, 452, 58 means $N = 452^2+3.58^2$:—

even 452, 58; 92, 262; 436, 90; 164, 250; 268, 218; 236,230; 428, 102; 412, 122;

odd 439, 85; 347, 177; 313, 197; 139, 255; 353, 173; 83,263; 457, 43; 293, 207; 461, 25; 193, 243; 463, 3; 227, 233; 367, 163; 61, 265; 389, 145; 23, 267.

N.B.—All the odds are obtainable from the evens by an obvious rule, and, since every P_{3m+1} produces the three forms $3A^2+B^2$,

Digitized by Google

$[\frac{1}{2}(3A \mp B)]^2 + 3[\frac{1}{2}(A \pm B)]^2$, one integral and two fractional, it follows immediately that the product of n similar primes equals 2^{n-1} integral and 2^n fractional values; consequently when N contains a factor 4 (or 4^m) the fractions disappear. In this case we get for the primes alone 8 integral and 16 fractional, and, multiplying by 4, we obtain by a second method the above 24 integral values.

(II.) It is known that a number N which has *several* partitions of form $(x^2 + 3y^2)$ must be *composite* and must consist of a set of factors $N = Eabcd\ldots$, where $E = 1$ or 4, and $a, b, c, d, \ldots$ are unequal *odd* primes of form $p = 6w + 1$. The number of such partition $(x^2 + 3y^2)$ of which N is capable will be 2^{r-1} if $E = 1$, or $3 \cdot 2^{r-1}$ if $E = 4$, where r = the number of *odd* primes. Here $24 = 3 \cdot 8 = 3 \cdot 2^{r-1}$, giving $r = 4$; so that $N = 4abcd$. And N will be least when a, b, c, d are the least *odd* primes of form $(6w + 1)$, *i.e.* 7, 13, 19, 31; therefore

$$N = 4 \cdot 7 \cdot 13 \cdot 17 \cdot 31.$$

Note on Question 14975.

By W. H. Blythe, M.A.

The cubic required is not $x(y^2 - z^2)\sin A + \ldots = 0$, as given in note to Question 10198, *Educational Times*, June, 1902, and *Mathematical Reprint*, Series II., Vol. II., p. 111, but

$$bcx(y^2 - z^2) + cay(z^2 - x^2) + abz(x^2 - y^2) = 0.$$

The tangents to the curve at the angular points and at the centre of gravity meet in the Lemoine point; the tangents at the centres of the inscribed and escribed circles meet in the centre of gravity.

Addition to Mr. Blythe's Note. *By* G. N. Bates.

This is the so-called "cubique des dix-sept points."

In addition to the fourteen points given in the question, the curve also passes through the mid-points of the perpendiculars. Vigarié in his *Géométrie du Triangle* gives references to Koehler "Sur un cubique remarquable du plan du triangle" (*Jour. de Math. Spec.*, 1887); and to Lemoine, *L'Annuaire de l'Association Française*, 1888; in addition to the reference Thomson, *Educational Times*, 1864.

10198. (John Griffiths, M.A.)—The coordinates x, y, z being proportional to the perpendicular distances from the sides of a triangle ABC, prove that (1) the locus of a point (x, y, z) such that its pedal circle (*i.e.*, the circle which passes through the feet of the perpendiculars

Digitized by Google

drawn from the point in question upon the sides of the triangle) touches the nine-point circle is the cubic

$$x(y^2-z^2)\cos A + y(z^2-x^2)\cos B + z(x^2-y^2)\cos C = 0;$$

and (2) trace this curve.

The PROPOSER gives the following interesting details as to the origin of the Question:—

The theorem in Question 10198 was derived by me from the following results.

(1) If P, P′; Q, Q′ denote two pairs of isogonal points, the sides of the triangle of reference and the lines PQ, PQ′, P′Q, P′Q′ are all tangents to a conic Σ.

(2) The director circle of Σ and the pedal circles of (P, P′), (Q, Q′) are coaxal.

(3) If a conic be drawn through the circumcentre to touch the sides, its director circle will touch the pedal of the circumcentre; *i.e.*, the nine-point circle.

Hence the theorem in question follows, if we suppose Q to coincide with the circumcentre and P, P′, Q to be collinear.

The asymptotes are parallel to the three lines OP_1, OP_2, OP_3, the joins of the circumcentre O with the three points P_1, P_2, P_3 (other than the vertices A, B, C) where the cubic intersects the circumcircle.

14915. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Show that

$$N = m^2\{(1-\nu)xy\}^4 + \nu\{x^2 - m(1+\nu)y^2\}^4$$

is factorisable. *Ex.*: $\frac{1}{4}N = 1 + \frac{1}{4}(xy)^4$ is factorisable when $x^2-3y^2=1$.

Solution by the PROPOSER.

Assume $N = (t^2+\nu u^2)(t'^2+\nu u'^2) = T^2 + \nu U^2 = m^2X^4 + \nu Y^4$;

therefore $N = (tt' - \nu uu')^2 + \nu(tu' + ut')^2$.

Take $t = Y+z$, $u = z$, $t' = z$, $u' = Y-z$; therefore

$$T = tt' - \nu uu' = (Y+z)z - \nu z(Y-z) = \{(1-\nu)Y + (1+\nu)z\}z,$$

$$U = tu' + ut' = (Y+z)(Y-z) + z^2 = Y^2.$$

Now take $z = (1-\nu)my^2$, $Y + m(1+\nu)y^2 = x^2$;

therefore $T = m\{Y + m(1+\nu)y^2\}\{(1-\nu)y\}^2 = m\{(1-\nu)xy\}^2$,

whence $N = m^2\{(1-\nu)xy\}^4 + \nu\{x^2 - m(1+\nu)y^2\}^4$.

Ex.—Take $m=1$, $\nu=2$; whence $N = 2 + (xy)^4$, if $x^2-3y^2=1$.

Here $z = -y^2$, $Y = 1$; $t = 1-y^2$, $u = -y^2-t'$, $u' = 1+y^2$; therefore

$$N = \{(1-y^2)^2 + 2y^4\}\{y^4 + 2(1+y^2)^2\}.$$

Ex. (1). $7^2 - 3.4^2 = 1$ gives $N = 2 + (7.4)^4 = 737.834$,

$26^2 - 3.15^2 = 1$ gives $N = 2 + (26.15)^4 = 151426.152777$.

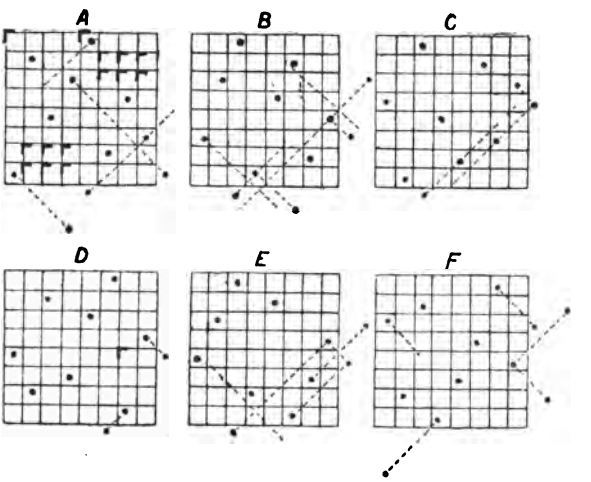
15021. (ALETROP.)—Ranger les nombres 1, 2, 3, ..., 8 de manière que la différence entre deux quelconques d'eux ne soit pas égale en valeur

Digitized by Google

absolue à la différence de leurs ranges. (Par ce problème on place sur l'échiquier huit *reines* en sorte qu'aucune d'elles ne donne échec à aucune autre.)

(I.) *Solution by* JAMES BLAIKIE, M.A.; (II.) *Table of Arithmetical Solutions by* H. W. WEBSTER, M.D.

(I.) By trial on the chess-board I find six entirely different solutions. These are shown in the diagram as A, B, C, D, E, F. Each of these solutions except E leads to eight different arithmetical solutions, as the chess-board may be looked at from four different sides and the columns may be taken either from left to right or *vice versa*. Thus A gives the eight solutions 17468253, 35286471, 83162574, 47526138, 64713528, 82531746, 52473861, 16837425. The solution E, however, only gives



four arithmetical solutions, as, owing to the symmetry of the figure, the configurations seen from opposite sides of the board are identical. If we place to the right of A a similar chess-board with the same arrangement of queens, and place a similar board below each, and employ a hollow frame enclosing sixty-four squares, we obtain new chess-board solutions except in those cases where two queens on the same diagonal are included. The diagonals which may contain two queens are dotted in the diagram. We now find that diagram A affords fourteen different chess-board solutions, the corner of the hollow chess-board being placed on the marked points. But seven of these solutions are reflections of the other seven, and thus we only obtain fifty-six arithmetical solutions. Diagrams B, C, E, F when treated in the same way do not afford any new solution; D affords two solutions, but, as they are reflections of each other, we have only eight arithmetical solutions. Thus the diagrams lead to ninety-two arithmetical solutions. It will be observed that the eight arithmetical solutions which, in general, correspond to one chess-

Digitized by Google

board solution may be derived from one another by easy arithmetical methods.

(II.) The following table shows twelve arrangements satisfying the conditions and each yielding subsidiary forms :—

1	2	3	4	5	6	7	8	
5	2	4	6	8	3	1	7	*a*
2	4	6	8	3	1	7	5	*b*
8	3	1	6	2	5	7	4	*c*
6	3	5	7	1	4	2	8	*d*
3	5	7	1	4	2	8	6	*e*
5	7	1	4	2	8	6	3	*f*
5	2	8	1	4	7	3	6	*g*
4	8	1	5	7	2	6	3	*h*
4	8	5	3	1	7	2	6	*i*
6	3	5	8	1	4	2	7	*j*
4	6	8	2	7	1	3	5	*k*
4	6	8	3	1	7	5	2	*l*

Form "*g*" was published in the *Illustrated London News* some years ago.

Note.—It will be found on examination that the arrangements *a*, *b*, *c*, *d*, *e*, *f*, and *l* in the above table are particular cases of diagram A or of one of its derived forms, and that the arrangements *g*, *h*, *i*, *j*, and *k* are particular cases of diagrams B, C, F, D, and E.

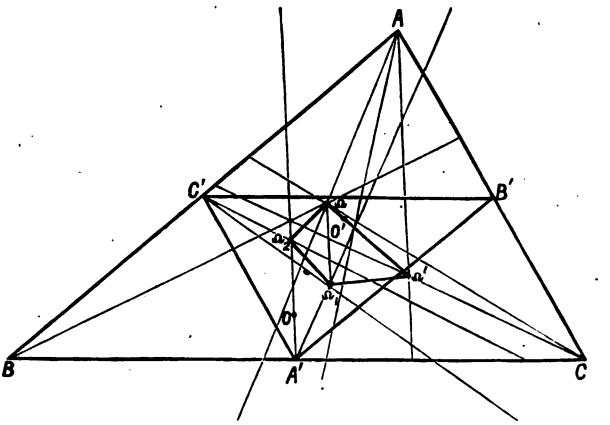
15092. (R. Tucker, M.A.)—ABC and A'B'C' are a triangle and its medial triangle. Ω, Ω'; Ω_1, Ω_2 are the Brocard points; and K, K' the symmedian points. Prove that $\Omega\Omega_1$, $\Omega'\Omega_2$ the join of the Brocard ellipse centres, and KK' pass through the common centroid. Also $\Omega\Omega_2$, $\Omega'\Omega_1$ intersect in $a^3\alpha = b^3\beta = c^3\gamma$. (*Cf.* Milne, *Companion*, p. 174, Ex. 11.)

Solution by the Proposer, *and* Professor Sanjána, M.A.

The a-coordinates of Ω, Ω'; Ω_1, Ω_2 are readily found to be proportional to ac^2, ab^2; $b^3c\,(c^2+a^2)$, $bc^3\,(a^2+b^2)$. Hence the equations to $\Omega\Omega_1$, $\Omega'\Omega_2$ are $\quad \Sigma ab^3\,(c^2-a^2)\,\alpha = 0, \quad \Sigma ac^3\,(a^2-b^2)\,\alpha = 0$ .....(i.).

Digitized by Google

Again, α-coordinates of BROCARD ellipse centres of ABC, A'B'C' are pro-



portional to $a(b^2+c^2) : bc(a^2b^2+b^2c^2+c^2a^2+b^2c^2)$; therefore equation to the join of these is $\Sigma a^3\alpha(b^2-c^2)=0$ .....(ii.). Hence (i.) and (ii.) these lines cointersect in G, the centroid of the triangles. Again, equations to $\Omega\Omega_2$, $\Omega'\Omega_1$ are

$$\Sigma a^3b^3\omega_c\alpha=0, \quad \Sigma a^3c^2\omega_b\alpha=0 \text{ .....(iii.)},$$

where $\omega_a = a^4-b^2c^2$; hence these lines meet in

$$a^3\alpha = b^3\beta = c^3\gamma.$$

Again, K, K' lie on $\Sigma a\alpha(b^2-c^2)=0$, which passes through G.

10157. (R. W. D. CHRISTIE.)—If p be a prime number >2, prove

$$\{(2p!/2\times p!\times p!)-1\} = p^2(\mathrm{M}),$$

where M is the sum of $\frac{1}{2}(p-1)$ integral square numbers.

Solution by the PROPOSER.

It is well known that the sum of the squares of the coefficients in the expansion of $(1+x)^n$, where n is a positive integer, is $2n!/n!\,n!$. Putting $x=1$, $n=p$ (a prime), then

$$2p!/p!\,p! = 1+p^2+(pq)^2+\ldots+(pq)^2+p^2+1;$$

$$\therefore \quad \frac{2p!}{2p!\,p!}-1 = p^2\left\{1+\left(\frac{p-1}{2!}\right)^2+\left(\frac{p-1\,.\,p-2}{3!}\right)^2+\ldots \text{ to } \frac{p-1}{2} \text{ terms}\right\}$$

$$= p^2(\mathrm{M}) = p^2\left\{\frac{2p-1!-p\,(p-1!)^2}{p^2\,(p-1!)^2}\right\}$$

$$= p^3(\mathrm{Q}), \text{ by WOLSTENHOLME, when } p>3.$$

In the *Journal of Pure and Applied Mathematics*, for October, 1861, the late Rev. Dr. WOLSTENHOLME says:—"The method I employed is somewhat laborious, and I should be glad if some of your readers would supply

Digitized by Google

a more direct proof" (*v.* also *Quarterly Journal*, June, 1899). The theorem gives the connexion between the integral squares and the series

$$\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n-1}\right).$$

How to reduce $\int \frac{dx}{\sin x+\sec x}$ *and* $\int \frac{dx}{\cos x+\operatorname{cosec} x}$ *to known Integrable Forms.*

By G. Scott, M.A.

For the sake of brevity let the first be called $f(x)$, the second $\mathrm{F}(x)$.

Then $f(x)=\int \frac{\cos x\,dx}{\sin x\cos x+1}$, $\mathrm{F}(x)=\int \frac{\sin x\,dx}{\sin x\cos x+1}$.

But $\sin x\cos x=\frac{1}{2}[(\sin x+\cos x)^2-1]$;

therefore $f(x)=2\int \frac{\cos x\,dx}{(\sin x+\cos x)^2+1}$;

again, $\sin x+\cos x=\sqrt{2}\{\sin(x+45°)\}$;

therefore $f(x)=2\int \frac{\cos x\,dx}{2\sin^2(x+45°)+1}=2\int \frac{\cos\{(x+45°)-45°\}\,dx}{2\sin^2(x+45°)+1}$,

or $f(x)=\sqrt{2}\int \frac{\cos(x+45°)\,dx}{2\sin^2(x+45°)+1}+\sqrt{2}\int \frac{\sin(x+45°)\,dx}{2\sin^2(x+45°)+1}$;

therefore $f(x)=\int \frac{d\{\sqrt{2}\sin(x+45°)\}}{\{\sqrt{2}\sin(x+45°)\}^2+1}-\int \frac{d\{\sqrt{2}\cos(x+45°)\}}{3-\{\sqrt{2}\cos(x+45°)\}^2}$,

$$f(x)=\int \frac{dx}{\sin x+\sec x}=\int \frac{du}{u^2+1}-\int \frac{dv}{3-v^2}.$$

Similarly, $\mathrm{F}(x)=\int \frac{dx}{\cos x+\operatorname{cosec} x}=-\int \frac{dv}{3-v^2}-\int \frac{du}{u^2+1}$,

where $u=\sqrt{2}\sin(x+45°)$ and $v=\sqrt{2}\cos(x+45°)$.

9781. If $ax+by+cz+dw=0$, and $a'x+b'y+c'z+d'w=0$, where a, b, c, d, a', b', c', d' are functions of a parameter θ, be the equations to a line, this line will generate a ruled surface of order $m+n$, where m and n are the orders of the two given equations as functions of θ. Especially examine the case when $m=n=1$, and show that a second set of lines xists in this case.

Solution by Professor Nanson.

The order is the number of generators which meet an arbitrary line $a''x+\ldots$, $a'''x+\ldots$, where a'', …, a''', … are constant, and is therefore equal to the order of the determinant $(abcd)$; that is, to $m+n$.

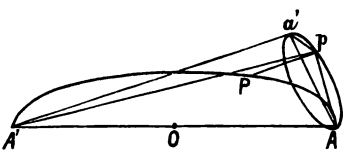
Digitized by Google

If $m = n = 1$, the generator is $u + v\theta$, $u' + v'\theta$, where u, v are linear in x, y, z, w, and the surface is therefore the quadric $uv' - u'v$. But this quadric is also generated by the line $u + u'\theta$, $v + v'\theta$.

9873. (J. O'Byrne Croke, M.A.) — If the sections of an ellipsoid through the axis of x be projected orthogonally into circles on planes passing through one of the extremities of that axis, those circles lie on the same sphere, and the envelope of their planes is the cone

$$\frac{y^2}{b^2(a^2-c^2)} + \frac{z^2}{c^2(a^2-b^2)} = \frac{(a-x)^2}{b^2c^2}.$$

Solution by A. M. Nesbitt, M.A.



Let APA′ be such a section; P any point on it; a', p the orthogonal projections of A′, P on the plane through A; so that, by given conditions, Apa' is a circle. Now A′a' is perpendicular to the plane of this circle, and therefore to Ap, which is in this plane; also Ap is perpendicular to pa' (angle in semicircle). Hence Ap is perpendicular to any straight line in plane containing A′a', pa'; therefore Ap is perpendicular to A′p. Thus A′pA is a right angle, or p always lies on a sphere with AA′ as diameter.

[This result is used in what follows: geometry is here easier than analysis.

Take for coordinates of P $(a\lambda, b\mu, c\nu)$ so that $\Sigma\lambda^2 = 1$; equation of plane APA′ is $y/b\mu = z/c\nu$; then plane of which envelope is required is

$$p(x-a) + c\nu y - b\mu z = 0,$$

p having to be determined by the conditions of the problem.

Now the equation of the right circular cylinder through A′ and the circle Apa' may be found to be

$$(px + c\nu y - b\mu z)^2 = a^2p^2 + (x^2+y^2+z^2-a^2)(p^2 + c^2\nu^2 + b^2\mu^2),$$

for it cuts the sphere $\Sigma x^2 = a^2$ in the two planes

$$p(x \pm a) + c\nu y - b\mu z = 0.$$

Expressing the condition that $(a\lambda, b\mu, c\nu)$ is on this cylinder, we get, on eliminating λ, $\quad p^2 = a^2(\mu^2+\nu^2) - b^2\mu^2 - c^2\nu^2$;
and, getting rid of p, we have

$$(a^2-b^2)\mu^2 + (a^2-c^2)\nu^2 = (c\nu y - b\mu z)^2.$$

The envelope got by expressing the condition that this is a perfect square in μ/ν is therefore the cone

$$\{(a^2-b^2)(x-a)^2 - b^2z^2\}\{(a^2-c^2)(x-a)^2 - c^2y^2)\} = b^2c^2y^2z^2,$$

which, on dividing by $(x-a)^2$, yields

$$(a^2-b^2)(a^2-c^2)(x-a)^2 = b^2z^2(a^2-c^2) + c^2y^2(a^2-b^2),$$

the result required.

Digitized by Google

15056. (Professor NEUBERG.)—Trouver le lieu du sommet d'un angle circonscrit à une parabole, les côtés de cet angle étant parallèles à deux diamètres conjugués quelconques d'une conique donnée.

Solution by J. H. TAYLOR, M.A., *and* Professor SANJÁNA, M.A.

If $ax^2 + 2hxy + by^2 = 1$ .....(1)

is the equation of a conic, and

$y = mx, \quad y = m'x$ .....(2, 3)

are the equations of conjugate diameters, then

$a + h(m + m') + bmm' = 0$.....(4).

If $y^2 = 4cx$ .....(5)

is the equation of a parabola, then

$y = mx + c/m$ and $y = m'x + c/m'$..... .....(6, 7)

represent a pair of tangents parallel to (2) and (3) respectively.

The coordinates of the intersection of (6) and (7) are

$x = c/mm', \quad y = c(m + m')/mm',$

whence, from (4), $ax + hy + bc = 0$.

15069. (ALETROP.)—N et k étant des entiers donnés, de combien de manières différentes a-t-on en nombres entiers $N = a^2 + kb^2$, lorsque cela soit possible? (Voyez GAUSS, *Disq. Arith.*, Lipsiæ, 1801, p. 219.)

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

Supposing N to be *expressible* in form $N = a^2 + kb^2$; let n = *number* of possible partitions.

Case i.—When k is +. Then n depends on the factors of N.

(1) If N be *prime*, then $n = 1$.

(2) If $N = N_1N_2$, where $N_1 \neq a_1^2 + kb_1^2$, $N_2 \neq a_2^2 + kb_2^2$, then $n = 1$.

(3) If $N = N_1N_2$, where $N_1 = a_1^2 + kb_1^2$ in n_1 ways, $N_2 \neq a_2^2 + kb_2^2$, then $n = n_1$.

(4) If $N = N_1N_2$, where $N_1 = a_1^2 + kb_1^2$ in n_1 ways, $N_2 = a_2^2 + kb_2^2$ in n_2 ways, then $n = 2n_1n_2$.

(5) If $N = N_1N_2N_3$, then n depends on n_1, n_2, n_3 in a similar way to the above.

Case ii.—When k is −, then the number of *different* (*i.e.*, non-automorphic) forms (n) of N follows the *same* rules as when k is +. And each such form yields an *infinity* of forms, its own *automorphs*, by multiplication by *any* power of the *unit-form* $(T^2 - kU)^\kappa$, where T, U are the least solutions of $T^2 - kU^2 = 1$ (but these automorphs are—for many purposes—*not to be considered distinct* expressions).

Digitized by Google

15096. (Robert W. D. Christie.)—Find two distinct sets of triangles equal in area and infinite in number, the respective products of whose circumscribed and inscribed radii are as the longest and shortest sides of one set.

Solution by the Proposer.

Let the sides of the two triangles be $7m$, $24m$, $25m$; and $13m$, $14m$, $5m$, respectively.

Then $r = \frac{S}{s} = \frac{84m^2}{28m} = 3m$; $R_1 = \frac{abc}{4S} = \frac{2730m^3}{84m^2 \times 4} = \frac{65m}{8}$;

$R = \frac{abc}{4S} = \frac{4200m^3}{336m^2} = \frac{25m}{2}$; $r_1 = \frac{S}{s} = \frac{84m^2}{21m} = 4m$.

Therefore $r : r_1 = 3 : 4$; $R : R_1 = 100 : 65$ or $20 : 13$.

Therefore $rR : r_1R_1 = 15 : 13$.

The general case is for sides a, b, c:—

A triangle.	*B triangle.*
$a = 2mn + m^2$,	$(4mn + 2m^2)\sqrt[4]{k}$,
$b = 2mn + 2n^2$,	$(4mn + 2m^2 + 1)\sqrt[4]{k}$,
$c = (m+n)^2 + n^2$.	$(4mn + 2m^2 - 1)\sqrt[4]{k}$.

where $$k = \frac{n(m+n)}{\left[3\{m(m+2n)\}^2 - 3\right]^4}.$$

In the above special case put $n = 3$, $m = 1$.

15114. (A. M. Nesbitt, M.A.)—If a, b, c be real and positive, prove that $a/(b+c) + b/(c+a) + c/(a+b) \geqq \frac{3}{2}$.

Solutions (I.) *by the* Proposer; (II.) *by* Rev. T. Roach, M.A., *and* J. Blaikie, M.A.; (III.) *by* T. Perle, A.I.A., *and* M. Peiris.

(I.) We have

$$(a+b)/(b+c) + (b+c)/(a+b) \geqq 2, \quad (b+c)/(c+a) + (c+a)/(b+c) \geqq 2,$$

$$(c+a)/(a+b) + (a+b)/(c+a) \geqq 2;$$

therefore $(2a+b+c)/(b+c) + (2b+c+a)/(c+a) + (2c+a+b)/(a+b) \geqq 6$;

therefore $2a/(b+c) + 2b/(c+a) + 2c/(a+b) \geqq 3$;

therefore, &c.

(II.) $a/(b+c) + b/(c+a) + c/(b+a) = \frac{3}{2} + \Sigma\left[a - \frac{1}{2}(b+c)\right]/(b+c)$

$$= \frac{3}{2} + \frac{\Sigma(2a-b-c)(c+a)(a+b)}{2(b+c)(c+a)(a+b)} = \frac{3}{2} + \frac{(ab+bc+ca)\Sigma(2a-b-c) + \Sigma a^2(2a-b-c)}{2(b+c)(c+a)(a+b)}$$

$$= \frac{3}{2} + \frac{\Sigma a^2(2a-b-c)}{2(b+c)(c+a)(a+b)} = \frac{3}{2} + \frac{\Sigma(a^2-b^2)(a-b)}{2(b+c)(c+a)(a+b)},$$

and no term of the numerator can be negative for real and positive values of a, b, c.

Digitized by Google

(III.) $\quad a/(b+c)+b/(c+a)+c/(a+b) \gtreqless \tfrac{3}{2},$

as $2\{a(c+a)(a+b)+b(b+c)(a+b)+c(b+c)(c+a)\}$

$$-3(b+c)(c+a)(a+b) \gtreqless 0,$$

i.e., as $2\{(a+b+c)(a^2+b^2+c^2)+3abc\}$

$$-3\{a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc\} \gtreqless 0,$$

i.e., as $\quad 2(a^3+b^3+c^3)-\{a^2(b+c)+b^2(c+a)+c^2(a+b)\} \gtreqless 0,$

i.e., as $\quad (a-b)(a^2-b^2)+(b-c)(b^2-c^2)+(c-a)(c^2-a^2) \gtreqless 0.$

Now, the factors of each term are either both of the same sign or zero.

Hence $\quad (a-b)(a^2-b^2)+(b-c)(b^2-c^2)+(c-a)(c^2-a^2) \not< 0.$

Hence $\quad a/(b+c)+b/(c+a)+c/(a+b) \not< \tfrac{3}{2}.$

14992. (Professor Neuberg.)—Par un point fixe A on mène une sécante quelconque qui coupe une courbe donnée Δ en B et une seconde courbe donnée Δ' en B′. Les tangentes menées en B à Δ et en B′ à Δ' se rencontrent en un point C. Construire la tangente en C de la courbe engendrée par ce point lorsque la sécante tourne autour de A.

Solution by Professor Nanson.

If the tangent at C to the locus of C cut BB′ in X, we have ultimately

$$BX : B'X = BC^2.\delta s/\rho : B'C^2.\delta s'/\rho',$$
$$BA : B'A = \delta s.\sin\phi : \delta s'.\sin\phi'$$
$$= \delta s.B'C : \delta s'.BC.$$

Hence the cross ratio (BB′, XA) is equal to

$$\rho' BC^2/\rho B'C^2 \quad \text{or to} \quad \rho' \sin^2\phi'/\rho \sin^2\phi.$$

15060. (Professor Morley.)—Prove that the circumcircle and nine-point circle of a triangle are inverse as to the director circle of the maximum inscribed ellipse.

Solution by Professor Sanjána, M.A.

Let O, G, N, H be respectively the circumcentre, centroid, nine-point centre, and orthocentre of a triangle ABC. If R be the circumradius, $\frac{1}{2}$R shall be the nine-point radius; and, if $OH = \beta$, then $ON = \frac{1}{2}\beta$, $OG = \frac{1}{3}\beta$, and $NG = \frac{1}{6}\beta$. Hence $OG : GN = R : \frac{1}{2}R$, so that G is a centre of similitude of the circum- and nine-point circles. Let OH meet the nine-point circle in P, Q, and the circumcircle in p, q; then for

Digitized by Google

centre G and radius $= (GP.Gp)^{\frac{1}{2}}$ or $(GQ.Gq)^{\frac{1}{2}}$, either circle will invert into the other. This radius is seen to have for its square

$$\tfrac{1}{3}(R^2-\tfrac{1}{9}\beta^2) = \tfrac{1}{3}[R^2-\tfrac{1}{9}R^2(1-8\cos A\cos B\cos C)]$$
$$= \tfrac{4}{9}R^2(1+\cos A\cos B\cos C).$$

The maximum inscribed ellipse must touch the sides at their middle points; hence its centre is at the centroid. Therefore the centre of the director circle is at G. The radius of the circle may be found from its trilinear equation thus. The equation of the ellipse is $\Sigma(a\alpha)^{\frac{1}{2}} = 0$, *i.e.*, $\Sigma(a^2\alpha^2) - 2\Sigma(ab\alpha\beta) = 0$.

Hence by the usual rule the director-circle is found to be

$$a\alpha^2\cos A + b\beta^2\cos B + c\gamma^2\cos C - 2(a\beta\gamma + b\gamma\alpha + c\alpha\beta) = 0$$

or $(a\cos A + \beta\cos B + \gamma\cos C)(a\alpha + b\beta + c\gamma) - 3(a\beta\gamma + b\gamma\alpha + c\alpha\beta) = 0$.

Therefore, if ρ be the radius [see Salmon, *Conics*, § 132, 6th ed.],

$$9\rho^2 = R^2\{9-6(\cos^2 A + \cos^2 B + \cos^2 C)$$
$$+ \cos^2 A + \cos^2 B + \cos^2 C - 6\cos A\cos B\cos C\}$$
$$= R^2\{9-5(1-2\cos A\cos B\cos C) - 6\cos A\cos B\cos C\}$$
$$= \tfrac{4}{9}R^2(1+\cos A\cos B\cos C).$$

N.B.—If the other centre of similitude H be taken, the second circle for which the circumcircle and nine-point circle shall be mutually inverse is that to which the triangle is self-polar.

15072. (R. Tucker, M.A.)—PSQ, RST are orthogonal focal chords of a parabola. O, O′ are the respective co-normal points (*i.e.*, PO, QO are normals, and also RO′, TO′). Show that the join of O, O′ passes through a fixed point, that O, O′ lie on the same parabola, and find the locus of the intersection of OO′ and the join of the third co-normal points of (P, Q) and (R, T).

Solution by Lionel E. Reay, B.A.

If P is the point m, Q is $-1/m$. Also, if R is m' and T is $-1/m'$, then, since PQ and PT are perpendicular, we have

$$\frac{4}{(m-1/m)(m'-1/m')} = -1 \quad \ldots\ldots (1).$$

Also, if O is (h, k), and the third co-normal point with P and Q is μ, then m, $-1/m$, μ are the roots of $am^3 + m(2a-h) - k = 0$;

therefore
$$0 = m - 1/m + \mu \quad \ldots\ldots (2).$$
$$(2a-h)/a = m(-1/m) + \mu(m-1/m) = -1-\mu^2 \quad \ldots\ldots (3),$$
$$k/a = m(-1/m)\mu = -\mu \quad \ldots\ldots (4);$$

therefore, from (3) and (4), (h, k) lies on $x/a = 3 + (y/a)^2$ or $y^2 = a(x-3a)$. Similarly O′ lies on this parabola.

Again, OO′ is $y-k = (k-k')/(h-h')(x-h)$;

that is, $y + a\mu = [-a(\mu-\mu')]/[a(\mu^2-\mu'^2)](x-3a-a\mu^2)$,

or
$$y + am = (-1)/(\mu+\mu')(x-3a-a\mu^2) \quad \ldots\ldots (5).$$

Digitized by Google

From (1) $4 = -\mu\mu'$; therefore there is one relation connecting the coefficients of (5); therefore it passes through a fixed point.

The line joining $\mu\mu'$ is $y - 2a\mu = 2/(\mu+\mu')(x - a\mu^2)$.

Subtracting from (5), we get

$$3a\mu = -3/(\mu+\mu')(x-a\mu^2) + 3a/(\mu+\mu');$$

therefore $$x = a(1-\mu\mu') = 5a;$$

therefore required locus is $x = 5a$.

15043. (Professor Cochez.)—Étudier les courbes

(1) $x^4 - x^3y + 4x^2 - xy + 1 = 0$, (2) $x^4y^2 - x^3 + 4x^2y^2 - x + y^2 = 0$.

Solution by A. M. Nesbitt, M.A.

(1) Curve is symmetrical in opposite quadrants. We have

$$y = x + 1/x + 2x/(x^2+1);$$

so that, if $x = 1$, $y = 3$. Thus $y > x$ (x positive). Also x and y always have same sign; $x = y$ is asymptote; also $x = 0$. If u denote the left-hand member of (1),

$$du/dx = 4x^3 - 3x^2y + 8x - y = 0, \quad \text{at } (1, 3);$$

$$du/dy = -x^3 - x, \quad \text{which is only zero when } x = 0.$$

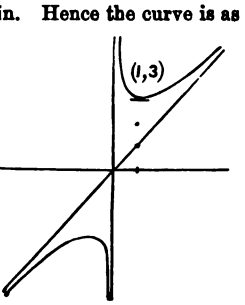
To discover the nature of the tangency of the y axis at infinity, introduce a third variable $z\,(=1)$ to make the equation homogeneous, and after put $y = 1$. This moves the x axis to infinity, and makes the line initially at infinity the new x axis ($z = 0$). Thus

$$x^4 - x^3 + 4x^2z^2 - xz^2 + z^4 = 0,$$

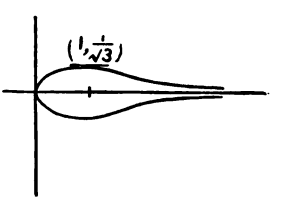
which has an ordinary tangent and an acnode at origin. Hence the curve is as represented, there being an acnode on the y axis at infinity.

(2) $$y^2 = x(x^2+1)/(x^4+4x^2+1).$$

Therefore $(1, \pm 1/\sqrt{3})$ are points on curve. If u denote the left-hand member of (2), $du/dx = 0$ at these points; while du/dy is only 0 if $y = 0$, which necessitates $x = \infty$. Further, x is never negative, and $y = 0$ is an asymptote, there being a cusp at infinity. There is a parabolic branch at origin. Hence the curve is as represented.



(1)



(2)

Digitized by Google

14955. (Aletrop.)—Donner la règle pour trouver tous les *ternes* de nombres consécutifs chacun desquels soit une somme de deux carrés entiers; *e.g.*: $72 = 6^2 + 6^2$, $73 = 8^2 + 3^2$, $74 = 7^2 + 5^2$.

Solutions (I.) *by* H. M. Taylor, M.A., F.R.S.; (II.) *by* J. H. Taylor, M.A.; (III.) *by* Lt.-Col. Allan Cunningham, R.E.

(I.) As every square number is of one of the forms $4n$, $4n+1$, or $4n+2$, it follows that, if each of three consecutive integers be the sum of two squares, the integers in ascending order of magnitude must be of the form $4n$, $4n+1$, and $4n+2$.

Let the three integers be $4x^2+4y^2$, $(2x+a)^2+(2y+b)^2$, and $(2x+c)^2+(2y+d)^2$. Then

$$4ax+4by+a^2+b^2=1 \quad \text{and} \quad 4cx+4dy+c^2+d^2=2.$$

Hence we see that the integers a and b are one even and one odd, of opposite sign, and prime to one another; that the integers c and d are both odd, of opposite sign, and prime to one another; and that the integers x and y are both even or both odd.

If any combination of numbers a, b, c, d, satisfying the relations just stated, be such that the above equations admit of an integral solution in x and y, then we obtain a set of three consecutive integers such as is required.

This is the case for all combinations of the numbers a, b, c, d which make $ad-bc=\pm 1$.

To find solutions, take b and c any two odd integers; take one of the two even integers $bc\pm 1$; then any pair of even and odd complementary factors of the number last selected may be chosen for a and d.

If the letters a, b, c, d receive the values 7, 4, 5, 3, respectively, we are led to the following solutions:—

$$32^2+48^2=3328, \quad 25^2+52^2=3329, \quad 27^2+51^2=3330;$$

and $160^2+272^2=99584$, $153^2+276^2=99585$, $165^2+269^2=99586$.

If $c=1$ and $d=-1$, then $x=y$.

This case leads to a series of solutions which may be written $2(n^2+n)^2$, $(n^2+2n)^2+(n^2-1)^2$, $(n^2+n+1)^2+(n^2+n-1)^2$, where n is any integer. The example given in the question is a particular case of this solution.

(II.) It is readily shown that (i.) the sum of two even squares is of the form $4n$; (ii.) the sum of one even and one odd square is of the form $4n+1$; (iii.) the sum of two odd squares is of the form $4n+2$.

It follows that, if the numbers are represented by

$$x^2+y^2=\mathrm{N}, \quad (x\mp a)^2+(y\pm b)^2=\mathrm{N}+1, \quad (x\pm c)^2+(y\mp d)^2=\mathrm{N}+2,$$

N is a multiple of 4; x, y, a are even, and b, c, d odd (all the letters represent integers).

(A) If $x=y^2$.—This leads to a series of solutions of the type

$$4n^4+4n^2=\mathrm{N}, \quad (2n^2+1)^2+(0)^2=\mathrm{N}+1, \quad (2n^2+1)^2+1^2=\mathrm{N}+2;$$

in each of which $(0)^2$ appears as the square of an even number, *e.g.*:

$$4\times 2^4+4\times 2^2=80, \quad 9^2+0^2=81, \quad 9^2+1^2=82.$$

Digitized by Google

(B) If $x = y$. — This leads to two series of solutions, of which the example given in the question is a particular case. Here, if

$$x^2 + x^2 = \mathrm{N}, \quad (x-a)^2 + (x+b)^2 = \mathrm{N}+1, \quad (x-1)^2 + (x+1)^2 = \mathrm{N}+2,$$

$x = \dfrac{a^2+b^2-1}{2(a-b)}$; the numerator is always an even digit, since a and b are one even and the other odd, and it is always divisible by $a-b$ if $a-b=1$. An example of the series in which a is odd: $a=3$, $b=2$ gives $x=6$; $6^2+6^2=72$, $3^2+8^2=73$, $7^2+5^2=74$.
Examples of the series in which a is even:

$$2^2+2^2=8, \quad 0^2+3^2=9, \quad 3^2+1^2=10;$$
$$12^2+12^2=288, \quad 8^2+15^2=289, \quad 13^2+11^2=290.$$

(C) If x and y are unequal.—

$$2(ax-by) = a^2+b^2-1 = \text{a multiple of } 4;$$
$$2(dy-cx) = c^2+d^2-1 = \text{a multiple of } 4.$$

These equations give integral values of x and y if

$$ad-cb = \pm 1.$$

Hence, either (i.) $a : b : c : d :: 2n : 2n+1 : 4n-1 : 4n+1$,
of which series the following is an example:—

$$(500)^2+(396)^2 = 406816, \quad (496)^2+(401)^2 = 406817,$$
$$(507)^2+(381)^2 = 406818;$$

or (ii.) $a : b : c : d :: 2n : 2n+1 : 4n+1 : 4n+3$,
of which series the following is an example:—

$$(2002)^2+(1722)^2 = 6973288, \quad (2008)^2+(1715)^2 = 6973289,$$
$$(1989)^2+(1737)^2 = 6973290.$$

Note.—The example given with the question is a particular case of the more general problem: Find the conditions that three integers in Arithmetical progression may be each the sum of two squares.
The following are examples of these series:—

(A′) $8^2+4^2=80, \quad 12^2+0^2=144, \quad 12^2+8^2=208.$

(B′) $(116)^2+(116)^2 = 26912, \quad (128)^2+(103)^2 = 26993,$
$(125)^2+(107)^2 = 27074.$

(C′) $(10972)^2+(11808)^2 = 259813648,$
$(10958)^2+(11821)^2 = 259813805,$
$(11001)^2+(11781)^2 = 259813962.$

(III.) Let x, y, z be the required numbers. As each is to be of form (a^2+b^2), the forms $(4i-1)$, $(4i+3)$ are inadmissible for them, and also for any factor of them, unless such factor be contained as a square. On examining the succession of numbers

$$(8i-1); \quad 8i; \quad 8i+1, 2, 3, 4, 5, 6, 7,$$

it is seen that $(8i-1)$, $(8i+3)$, $(8i+6) = 2(4i+3)$, $8i+7$ are inadmissible; so that the only admissible *triad* is

$$x = 8i, \quad y = 8i+1, \quad z = 8i+2 \quad \ldots\ldots\ldots\ldots\ldots (1).$$

Digitized by Google

Now, taking $i = n^2$ gives

$$x = (2n)^2 + (2n)^2, \quad y = 8n^2 + 1, \quad z = (2n-1)^2 + (2n+1)^2 \ldots\ldots \quad (2),$$

wherein x, z are of required form; and y will also be the proper form if

$$y = \text{prime, or product of primes each of form } (4w + 1) \ldots\ldots \quad (3).$$

Now $n = 3\nu \pm 1$ gives $y = 8(9\nu^2 \pm 6\nu) + 9$, which contains the factor 3, but not 9—an impossible form unless $\nu = 3\nu'$. Hence $n = 3\nu$ and $n = 9\nu' \pm 1$ are possible forms (but with exceptions).

Ex.—$\nu = 1$ gives $n = 3$;
$x = 6^2 + 6^2 = 72$, $y = 73 = 3^2 + 8^2$, $z = 5^2 + 7^2 = 74$.
$\nu = 2$ gives $n = 6$;
$x = 12^2 + 12^2 = 288$, $y = 289 = 15^2 + 8^2$, $z = 11^2 + 13^2 = 290$.
$\nu = 3$ gives $n = 9$; $y = 649 = 11 \cdot 59$—an impossible form.
$\nu' = 1$ gives $n = 10$;
$x = 20^2 + 20^2 = 800$, $y = 801 = 15^2 + 24^2$, $z = 19^2 + 21^2 = 802$.
$\nu' = 2$ gives $n = 17$;
$x = 34^2 + 34^2 = 2312$, $y = 2313 = 3^2 + 48^2$, $z = 33^2 + 35^2 = 2314$.

Here (2) and (3) give a pretty general solution; but it does not seem easy to prove it to be *complete*.

9309. (Professor Hudson, M.A.)—A cage, mass M, is drawn up a mine by an engine, at first with uniform acceleration, then with uniform velocity v, and then for a distance equal to the first portion with uniform retardation; the whole time of ascent is t, and the greatest horse-power at which the engine is worked is H. Prove that the depth of the mine is

$$\frac{550\mathrm{H}gvt - \mathrm{M}v^2(gt + v)}{(550\mathrm{H} - \mathrm{M}v)\,g}.$$

Solution by Professor Sircom.

Let t_1 be the time of the accelerated motion; then, by the question,

$$x = \tfrac{1}{2}vt_1 + v(t - 2t_1) + \tfrac{1}{2}vt_1,$$

and

$$vt_1 = vt - x.$$

Let the mass of the cage be M *pounds*. The greatest horse-power will be exerted when the velocity is on the point of reaching v. Then the power is equal to the tension of the chain multiplied by the velocity, or $\mathrm{T}v = 550\mathrm{H}g$, in absolute units, and $\mathrm{T} = 550\mathrm{H}g/v$, and the work done during the period of acceleration is $550\mathrm{H}g/v \cdot \tfrac{1}{2}vt_1$; this is equal to the work done in raising the weight together with that done in communicating the velocity v. Hence

$$550Hg/v\,\tfrac{1}{2}(vt - x) = \mathrm{M}g\,\tfrac{1}{2}(vt - x) + \tfrac{1}{2}\mathrm{M}v^2,$$

whence

$$x = \frac{550\mathrm{H}gvt - \mathrm{M}v^2(gt + v)}{(550\mathrm{H} - \mathrm{M}v)\,g}.$$

Digitized by Google

14987. (T. Muir, M.A., F.R.S.)—Given

$$u \equiv (a, b, c, d\,\text{Q}\,x, y)^3 + e = 0,$$

show that $$\left(\frac{d^2y}{dx^2}\right)\left(\frac{du}{dx}\right)^3 = 2e\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ y^3, & -xy, & x^2 \end{vmatrix},$$

and generalize.

Solution by Professor Nanson.

Denoting the determinant in the question by Δ, we have, using Salmon's notation,

$$\left(\frac{d^2y}{dx^2}\right)\left(\frac{du}{dy}\right)^3 = \begin{pmatrix} u \\ u \end{pmatrix} = \frac{3e}{2}\,\mathrm{H} = 54e\Delta,$$

since H is the Jacobian of the quadrics u_x, u_y, and therefore equal to 36Δ.

14999. (R. Chartres.)—If the in-circle of a triangle passes through the centroid, (1) show that $5\Sigma(a^2) = 6\Sigma(ab)$ (Question 12641); (2) give an easy method of writing down in integers the sides of any number of such triangles; and (3) find the mean value of the in-radius as the smallest side takes all possible values between 1 and 2 units.

Solution by the Proposer.

(1) This easily follows since r = the distance between the two points, the in-centre and centroid.

(2) It is satisfied by $a = n^2+1$, $b = (n+1)^2+1$, $c = n^2+(n+1)^2$, n being different from zero.

If $n = 1, 2, 3, \ldots$, the sides are (2, 5, 5), (5, 10, 13), (10, 17, 25), ...

If n be fractional, we get other series of values.

(3) $$r = \frac{\Delta}{s} = \frac{n(n+1)}{\sqrt{2}\,\sqrt{(n^2+n+1)}};$$

therefore mean value of r, when the smallest side lies between 1 and 2,

$$= \frac{1}{\sqrt{2}}\int_0^1 \left(\sqrt{(n^2+n+1)} - \frac{1}{\sqrt{(n^2+n+1)}}\right) dn$$

$$= \frac{1}{\sqrt{2}}\left\{\frac{3\sqrt{3}-1}{4} - \tfrac{1}{2}\log\left(\frac{3+2\sqrt{3}}{3}\right)\right\}.$$

Note by Professor Sanjána, M.A.

In Question 12641 I first drew attention to the analytical condition given above. It is readily obtained by taking the in-circle in trilinear co-ordinates, $\Sigma\sqrt{(a\cos^2\frac{1}{2}A)} = 0$, and making the centroid fall upon it; we obtain $\Sigma\sqrt{(\cos^2\frac{1}{2}A + a)} = 0$, which gives $\Sigma\sqrt{(s-a)} = 0$. On rationalization this becomes $5s^2 = 4(ab+bc+ca)$, *i.e.*, $5\Sigma(a^2) = 6\Sigma(ab)$.

An easy way, and the most natural one, of obtaining such triangles is to give one side a numerical value; an indeterminate equation of the second degree between the remaining sides results, which can be solved

Digitized by Google

by ordinary algebraical methods. Without loss of generality, we can take $a = 1$; then $b = \frac{1}{3}\{3(c+1) \pm 4\sqrt{(3c-c^2-1)}\}$.

If the value of b is fractional, multiply out by its denominator, and the sides may be written down in integers. The process thus depends on solving the diophantine equation $c^2-3c+1+\square = 0$. The only isosceles solutions are the line-triangle 1, 1, 2 and the triangle 2, 5, 5; and by various artifices all other triangles may be made to depend on these, or rather on the first. Thus among many series of such triangles I have found those in the margin. They are obtained thus: Write down the line-triangle 1, 2, 1; take 2 for the smallest side of the next, and the other sides of this are the remaining sides of the first increased by 4; taking this triangle, make 5 the smallest side of the next, and the other sides of this third are the remaining sides of the second increased by 8; and so on. The increments are seen to be the terms of 4, 8, 12, 16, 20, ... for the several derivative triangles. It will be found that the excess of any two sides of a triangle of this series over the third is of the form $2p^2$ (a theorem generally true); and these excesses are in order $2(1^2, 0^2, 1^2)$; $2(1^2, 1^2, 2^2)$; $2(1^2, 2^2, 3^2)$; $2(1^2, 3^2, 4^2)$; and so on.

1	2	1
2	5	5
5	10	13
10	17	25
17	26	41
26	37	61
37	50	85
50	65	113
65	82	145
...	...	...

Another curious series of such triangles which I get is shown in the margin. Beginning with the line-triangle now written as 1, 1, 2, we take 2 for the smallest side of the next, and increase the others by 4 to get the remaining ones; of 2, 5, 5, we take 5 for the smallest side of the next, and increase the others by 8 to determine the third; next, taking 13 for the smallest side of the fourth, we increase the others by 24, and thus determine this triangle; and so on. The increments form the series 4 (1, 2, 6, 15, 40, ...) for the several derivative triangles; the terms in the bracket are the products of two contiguous terms of the series 1, 1, 2, 3, 5, 8, 13, ... The excesses are found to be $2(0^2, 1^2, 1^2)$; $2(1^2, 2^2, 3^2)$; $2(2^2, 3^2, 5^2)$; $2(3^2, 5^2, 8^2)$; and so on.

1	1	2
2	5	5
5	10	13
13	29	34
34	73	89
89	194	233
...	...	...

We suppose above the increments to be equal; but any other supposition can be made. For instance, if one set of values a', b', c' be known, we may assume a second set to be a', $b'+mx$, $c'+nx$, where m and n are any fractional or integral positive or negative quantities. This gives for x a quadratic one of whose roots is zero (thus giving the original triangle); the other root will give a new triangle. Thus, if a', b', c' be 29, 13, 34, and $m = 2$, $n = 1$, we find $x = 31\frac{5}{13}$; hence the derived triangle is 29, $75\frac{10}{13}$, $65\frac{5}{13}$, or, in integers, 377, 850, 985. If we take 26, 37, 61, and make the same changes, we get $x = 5\frac{7}{13}$; the new triangle is 26, $48\frac{1}{13}$, $66\frac{7}{13}$, or, in integers, 338, 625, 865. This process gives an infinity of derived triangles from any specified triangle.

Again, as the condition may be written $5(\Sigma a)^2 = 16(\Sigma ab)$, the sum of the sides will necessarily be a multiple of 4; and two sides will, in

Digitized by Google

general, but not necessarily, be multiples of 5. Hence by multiplying two sides by 5, or dividing out by it where possible, new triangles can easily be formed. For instance, take 377, 850, 985 found above; make 170 (850/5) and 197 (985/5) the sides b, c of a new triangle, and find a; the quadratic $5a^2 - 2202a + 137605 = 0$ is obtained. One factor of the left side must be $5a - 377$, as is seen from the original triangle; hence the other factor is $a - 365$, and we get the new triangle to be 170, 197, 365. Next take 34, 73 as two sides of a triangle derived from this; the third will be found to be 89; this triangle is found in the second marginal series. Similarly, from the triangle 338, 625, 865 above we derive successively 125, 290, 173; 25, 58, 65; 5, 10, 13; and, finally, 1, 1, 2. The converse process is also useful in the derivation of such triangles. In fact the field of the curious properties possessed by these triangles seems very extensive.

15087. (Professor E. B. Escott.)—In *Nature* for May 6, 1897, we have the following "Sieve for Primes" given:—"In order that a number n be prime, it is necessary and sufficient that the $\frac{1}{2}(n+1)$-th term of the following recurrent series give a remainder 1 when divided by n:—

1, 4, 11, 29, 76, 199, 521, ...,

where the scale of relation is $u^n = 3u_{n-1} - u_{n-2}$." Show, by a numerical example, that this condition is not sufficient.

Solution by the Proposer.

If we consider the recurring series 1, 3, 4, 7, 11, 18, 29, ..., where the scale of relation is $V_n = V_{n-1} + V_{n-2}$, the $(2n+1)$-th term of this series will be the $(n+1)$-th of the series considered by Mr. Christie, and the theorem will be that V_n gives a remainder 1 when divided by n.

It is easily proved by taking residues of the terms to the modulus 3, 5, 7, ... that $V_n \equiv 1 \pmod 3$, when $n \equiv 1, 3$, or $4 \pmod 8$;
$V_n \equiv 1 \pmod 5$, when $n \equiv 1 \pmod 4$;
$V_n \equiv 1 \pmod{47}$, when $n \equiv 1, 15 \pmod{32}$.

If we let $n = 3 \times 5 \times 47 = 705$, since 705 is $\equiv 1 \pmod{32}$, and therefore $\pmod 4$ and $\pmod 8$, $V_{705} \equiv 1 \pmod{705}$, which disproves the theorem, since 705 is not a prime number.

A test somewhat like this was proposed in Question 1484, *L'Intermédiaire des Mathématiciens*, 1899, p. 76. Although it seems probable that the test is not sufficient, it has not been disproved. In the series

0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, ...,

where the law of recurrence is $W_n = W_{n-2} + W_{n-3}$, in order that n be a prime number, it is necessary and sufficient that $W_n \equiv 0 \pmod n$.

Remarks by R. W. D. Christie.

The "sieve" was discovered many years ago in the course of some investigations into the properties of the roots of $x^5 \pm 1 = 0$. I soon

Digitized by Google

found that the "continuant" series was the real prime series, *e.g.*:

$$\frac{(\omega_4+\omega_5)^n-(\omega_2+\omega_3)^n}{(\omega_4+\omega_5)-(\omega_2+\omega_3)} = 1, 1, 2, 3, 5, \ldots.$$

There are other series besides from which the primes are recoverable.

Let a, b be any two consecutive elements of the series; then it is not difficult to establish $a^2-b^2+4ab-1 = P(Q)$, where P is any prime; and, since $a^2-b^2-ab = \pm 1$, it follows that $5ab = P(Q)$ if P be of form $4m+1$, and $5ab-2 = P(Q)$ when of form $4m-1$. Consequently, P_{4m+} is a factor of either a or b, *e.g.*:

$$a_n = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots;$$
$$\text{primes} = 1, 3, 5, 7, (9), 11, 13, (15), 17, 19, \ldots.$$

Here for 13 we have $8, 13 = b, a$;

therefore $5ab = P(Q)$, since $13 = 4m+1$;

and for 11 we have $5, 8 = b, a$;

therefore $5ab-2 = P(Q)$, since $11 = 4m-1$.

Another series of the same kind as given by me in *Nature* is

$$a_n = 1, 7, 41, 239, 1393, 8119, 47321, 275807, 1607521, 9369319, \ldots$$
$$\text{primes} = 1, 3, 5, 7, (9), 11, 13, (15), 17, 19, \ldots.$$

Of course, in the example mentioned, we should not, in practice, think of using the "sieve" for any number ending in 5. It is, I find, a complete exception which apparently acts as a quasi-prime and does not vitiate the theorem.

Incidentally, I may mention the integral primitive roots of every prime have their origin in the above "continuant" series, and can be proved to be composite.

15130. (Professor Langhorne Orchard, M.A., B.Sc.)—An equilateral triangle, of side-length = a, is thrown "at random," from a height = na, upon an indefinite horizontal plane which is ruled with equidistant horizontal lines, crossed with equidistant vertical lines, so as to form a number of squares, each of side-length $2a$. Find the "probability" that a side of the triangle will cut one of the lines.

Solution by D. Biddle.

Assuming the height from which the triangle falls to be sufficient to render the position of the triangle, as it lies on the floor, "random," it would appear that, if p = the chance that the figure lies between two parallels of one set, p^2 = the chance that it will lie wholly within a square, and $1-p^2$ = the chance that it will lie across a line of one or both sets. The last-mentioned chance may be broken up as follows:—$p-p^2$ = the chance that it lies across one side only, and that belonging to a particular set of parallels; $2(p-p^2)$ = the chance that it lies across

Digitized by Google

one side only, and that irrespective of the set of parallels to which it belongs; $1-2p+p^2$ = the chance that it lies across two lines (one of each set). The sum of the last two chances exactly makes $1-p^2$.

By reference to Question 10219 (Professor SYLVESTER) and the italicized portion of my solution of it on p. 98, Vol. LII., the following law will be found enunciated (not for the first time):—"The ratio borne to the periphery of the given circle (touching two adjacent parallels) by the length of a string drawn tightly round any plane figure whatsoever (capable of lying in all positions between the parallels) is invariably the probability that such figure will be cut by a parallel, if disposed at random on a ruled floor." Consequently,

$$1-p = 3/(2\pi) \quad \text{and} \quad P = 1-p^2 = (12\pi-9)/(4\pi^2) = P_1+P_2,$$

where $P_1 = 2(p-p^2) = (6\pi-9)/(2\pi^2)$ and $P_2 = 1-2p+p^2 = 9/(4\pi^2)$.

Numerically, the values are nearly as follows:—

$$P_1 = \cdot 498984, \quad P_2 = \cdot 227973, \quad P = \cdot 726957.$$

The chance that the triangle shall lie wholly within a square is ·273043.

15103. (J. S. LAWSON, M.A.)—Show that the sum of the harmonic series $\frac{1}{a}+\frac{1}{a+b}+\frac{1}{a+2b}+\ldots$ to n terms $> \frac{2n}{2a+(n-1)b}$.

Solutions (I.) *by* Professor SANJÁNA, M.A., A. M. NESBITT, M.A., *and* A. W. POOLE, B.A.; (II.) *by* CONSTANCE I. MARKS, B.A.

(I.) α and β being real and positive, $(\alpha+\beta)^2 > 4\alpha\beta$, and hence we have

$$1/\alpha + 1/\beta > 4/(\alpha+\beta).$$

Hence, when n is even, and a and b real and positive,

$$\frac{1}{a}+\frac{1}{a+(n-1)b} > \frac{4}{2a+(n-1)b}, \quad \frac{1}{a+b}+\frac{1}{a+(n-2)b} > \frac{4}{2a+(n-1)b}, \ldots.$$

Thus $\frac{1}{a}+\frac{1}{a+b}+\frac{1}{a+2b}\ldots$ to n terms $> \frac{4\times\frac{1}{2}n}{2a+(n-1)b} > \frac{2n}{2a+(n-1)b}$.

When n is odd the middle term will be $\frac{1}{a+\frac{1}{2}(n-1)b}$ or $\frac{2}{2a+(n-1)b}$: the remaining $n-1$ terms may be shown greater than $\frac{2(n-1)}{2a+(n-1)b}$.

(II.) All the terms of the series being first considered not only real but positive, if $n=2$, the theorem to be proved is

$$\frac{1}{a}+\frac{1}{a+b} > \frac{4}{2a+b}, \quad (2a+b)^2 > 4a(a+b), \quad b^2 > 0 \quad \text{(which is true)}.$$

If $n=3$, the theorem to be proved is

$$\frac{1}{a}+\frac{1}{a+b}+\frac{1}{a+2b} > \frac{6}{2(a+b)} \quad \text{or} \quad \frac{3}{a+b},$$

that is $(a+b)(a+2b)+a(a+2b)+a(a+b) > 3a(a+2b), \quad 2b^2 > 0$

Digitized by Google

(which is true). Suppose the law holds for all integral values of n from 2 up to r inclusive, so that

$$\sum_{0}^{r-1} \frac{1}{a+sb} > \frac{2r}{2a+(r-1)b}.$$

Then $\sum_{0}^{r} \frac{1}{a+sb}$ is certainly greater than $\frac{2(r+1)}{2a+rb}$, if

$$\frac{2r}{2a+(r-1)b} + \frac{1}{a+rb} > \frac{2(r+1)}{2a+rb},$$

if $2(a+rb)(2a+rb) + \{2a+(r-1)b\}(2a+rb)$

$$> 2(r+1)(a+rb)\{2a+(r-1)b\},$$

which readily reduces to the condition $r(r+1)b^2 > 0$ (which is true).

Hence, if the theorem holds up to $n = r$, it will also hold for the next value of n. But it does hold when $n = 2$ and when $n = 3$; therefore it is true when $n = 4$, and so on universally.

If the terms of the series are all negative, the sign of the inequality must be reversed. The case in which at some point in the given series the sign of the constituent terms changes may be regarded as compound, the above results applying to the two portions considered separately.

15110. (Professor SANJÁNA, M.A.)—Prove that the solution in rational numbers (integral or fractional) of the equation $M^2 - 2xN^2 = x^2 - 1$ will serve to factorise quantities of the form $N_1^4 + N_2^4$; and show that x, supposed a positive integer, will always be of the form $4q^2 + 1$.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

Let $F = N_1^4 + N_2^4$ be the number to be factorised. Assume

$$F = (N_1^2 + x \cdot N_2^2)^2 - (N_2 \cdot z)^2 \quad \text{.....(A)},$$

a *difference of squares*, and therefore directly factorisable. This requires

$$z^2 = 2x \cdot N_1^2 + (x^2 - 1) N_2^2 \quad \text{.....(B)}.$$

Writing $z/N_2 = M$, $N_1/N_2 = M$, and rearranging, reduces this to the proposed equation $M^2 - 2x \cdot N^2 = x^2 - 1$ .....(B′).

Evidently every integer solution of (B), or every integral or fractional solution of (B′), yields values of F along with their reduction to a difference of squares.

This method of factorisation was introduced by the present writer to the London Mathematical Society (see *Proceedings*, Vol. XXXII., p. 1) in a verbal communication. It was then shown that *every* number F of above form yields several (at least four) values of x, which may be + or −, and integral or fractional. *Each* such value of x leads to an *easily* solvable diophantine equation of type (B) or (B′), termed the *factorisant*, which has an infinite number of integral solutions. Each such solution (N_1, N_2) gives a *factorisable* number of form F, with the *value of z needed for its factorisation*.

Digitized by Google

Ex.— Take $F = 17 = 1^4 + 2^4 = 9^2 - 8^2 = n_1^4 + n_2^4$.

Here (1), $n_1 = 1$, $n_2 = 2$; then, by (A), $n_1^2 + x \cdot n_2^2 = \pm 9$, whence

$$x = \tfrac{1}{4}(\pm 9 - 1) = 2 \text{ or } -\tfrac{5}{2},$$

or (2), $n_1 = 2$, $n_2 = 1$; then, by (A), $n_1^2 + x \cdot n_2^2 = \pm 9$, whence

$$x = \pm 9 - 4 = 5 \text{ or } -13.$$

Thus the prime F = 17 has yielded *four* suitable values of x, *each* of which will yield a *solvable* factorisant (B).

Thus, selecting the value $x = 2$, the factorisant (B) is $(2N_1)^2 + 3N_2^2 = z^2$. An interesting train of solutions is obtainable with N_1 = constant = 1; then, by known methods,

$$N_2 = 2, \quad 8, \quad 30, \quad 418, \quad 1560, \text{ \&c.};$$

$$z = 4, \quad 14, \quad 112, \quad 724, \quad 2702, \text{ \&c.}$$

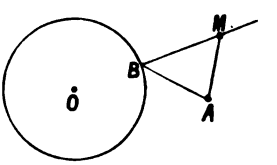
Thus $F = 1^4 + 1560^4 = (1^2 + 2 \cdot 1560^2)^2 - (1560 \cdot 2702)^2 = 652081\,(313 \cdot 29017)$.

Note that the base number (F = 17) above, being *prime*, has yielded only four suitable values of x. A composite base number (F) would yield *four* values of x *for every way* in which it could be expressed as a difference of squares.

[The above example shows that, even if x be a positive integer, it is not necessarily of form $x = 4q^2 + 1$, as stated in the Question; but, to give the full advantage of the method (which is really one of considerable power), x should not be limited to being either positive or integral. For another application of the same process see the present writer's Solution of Question 14971. The present writer has further used this method with great success for factorisation of other quartic forms, *e.g.*,

$$F = N_1^4 - N_1^2 N_2^2 + N_2^4, \text{ \&c.}]$$

15017. (Professor Cochez.)—On donne une circonférence O et un point fixe A. On joint A à un point B de la circonférence, et sur AB on construit un triangle ABM semblable à un triangle donné. Lieu de M quand B décrit la courbe.



Solutions (I.) *by* J. Prescott, B.A., H. L. Trachtenberg, *and others*; (II.) *by* Lionel E. Reay, B.A., *and* A. W. Poole, B.A.

(I.) Since ABM is similar to a given triangle, AM/AB = a constant; Also ∠ BAM is constant. Take a point P along AM, making AP = AB. Then as B describes the given curve P describes a similar and equal curve, which could be obtained by rigidly connecting the given curve with A and then rotating it about A through an angle = ∠ BAM.

Now, since AM/AP = a constant, M describes a curve similar to the one described by P, and therefore similar to the given curve. Hence, if the given curve is a circle, M describes a circle.

Digitized by Google

(II.) If AO = c, OB = a, AB = r, BAO = θ, BAM = β, MBA = α, then $a^2 = r^2 + c^2 - 2cr\cos\theta$. If AM = ρ, MAO = $\beta + \theta = \phi$, say, then $\rho \sin(\beta+\alpha) = r \sin\alpha$; therefore

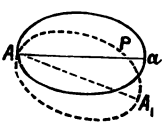
$$a^2 = \rho^2[\sin^2(\beta+\alpha)/\sin^2\alpha] + c^2 - 2c\rho[\sin(\beta+\alpha)/\sin\alpha]\cos(\phi-\beta),$$

or $a^2\lambda^2 = \rho^2 + c^2\lambda^2 - 2c\lambda\rho\cos(\phi-\beta)$, where $\lambda = \sin\alpha \operatorname{cosec}(\alpha+\beta)$; therefore locus of M is a circle whose centre is the point $(c\lambda, \beta)$ and radius $a\lambda$.

15055. (Rev. T. Roach, M.A.)—If two ellipses have equal major axes and a common vertex, show that they meet in only one other point.

Solution by the Proposer.

Let Aa, AA_1 be axes. Then every chord AP of the lower curve is < AA_1; therefore < Aa; therefore above Aa the lower curve lies inside the upper, and, since AA_1 > AP, it crosses between a and A_1. Similarly, below A_1 the upper curve lies inside the lower; therefore the curves cut in only one other point.



15002. (R. Knowles, B.A.)—From a point T (h, k), tangents TP, TQ are drawn to meet the parabola $y^2 = 4ax$ in P and Q. Prove that (1) the coordinates of the symmedian point of the triangle TPQ are

$$[(a+h)(k^2+2ah),\ k(k^2+2ah+6a^2)]/2(k^2-ah+3a^2);$$

(2) if PQ be a focal chord, this point is on the tangent at the vertex.

Solutions (I.) *by* R. Tucker, M.A.; (II.) *by* Lionel E. Reay, B.A.

(I.) (1) The following results are easily got, if we suppose the parabola to be $y^2 = 4x$. Equation to PQ,

$$ky - 2x - 2h = 0 \quad \text{.....(i.)};$$

therefore perpendicular from T is $(k^2-4h)/\sqrt{(k^2+4)}$, and

$$PQ = \sqrt{(k^2+4 \,.\, k^2-4h)};$$

hence

$$2\Delta = (k^2-4h)^{\frac{3}{2}},$$

also

$$\Sigma(PQ)^2 = 2(k^2-4h)(k^2-h+3);$$

therefore perpendicular from symmedian point on PQ

$$= \frac{\sqrt{(k^2+4)}\,(k^2-4h)}{2(k^2-h+3)} \quad \text{.....(ii.)}.$$

But this point lies on ST .....(A);

therefore, if (X, Y) are its coordinates, we have perpendicular from it on PQ

$$= (kY - 2X - 2h)/\sqrt{(k^2+4)} \quad \text{.....(iii.)}.$$

Digitized by Google

From (ii.) and (iii.) after reduction, we get

$$kY - 2X = \frac{k^4 + 2k^3 - 4h - 4h^2}{2(k^2 - h + 3)} \quad \text{.....(iv.)},$$

also, from (A), $kX - (h-1)Y = k$ .....(v.);

hence $(k^2 - 2h + 2)X = \frac{k^4 + 2k^2h + 2k^3 + k^4h + 4h - 4h^2}{2(k^2 - h + 3)}$,

i.e., $X = (k^2 + 2h)(1 + h)/2(k^2 - h + 3)$,

i.e., $X = (k^2 + 2ah)(a + h)/2(k^2 - ah + 3a^2)$ for $y^2 = 4ax$.

Hence Y can be verified.

(2) is obvious, since $\angle T$ is in this case a right angle.

From the above result we can at once answer my Question 14774 (August, 1901), of which a solution is given on p. 325 *Ed. Times*.

(II.) (1) If P and Q are the points m_1, m_2, then

$$h = am_1m_2, \quad k = a(m_1 + m_2) \quad \text{.....(i., ii.)}.$$

Now the multiples for which K, the symmedian point, is the mean centre of TPQ are PQ^2, QT^2, TP^2. Then, if K is the point (X, Y),

$$X = \frac{PQ^2 \cdot h + QT^2 \cdot am_1^2 + TP^2 \cdot am_2^2}{PQ^2 + QT^2 + TP^2}.$$

Writing down the values of PQ^2, &c., in terms of h, k, m_1, m_2, and then using (i.) and (ii.), the denominator becomes

$$2/a^2(k^2 - 4ah)(k^2 - ah + 3a^2),$$

and the numerator $1/a^2(k^2 - 4ah)(a + h)(k^2 + 2ah)$;

therefore $X = \frac{(a + h)(k^2 + 2ah)}{2(k^2 - ah + 3a^2)}$,

also $Y = \frac{PQ^2 \cdot k + QT^2 \cdot 2am_1 + TP^2 \cdot 2am_2}{PQ^2 + QT^2 + TP^2}$.

The numerator becomes $k/a^2(k^2 - 4ah)(k^2 + 2ah + 6a^2)$;

therefore $Y = \frac{k(k^2 + 2ah + 6a^2)}{2(k^2 - ah + 3a^2)}$.

(2) If PQ is a focal chord, $h = -a$; therefore $X = 0$; therefore K lies on the tangent at the vertex.

15144. (R. Knowles.)—Prove that the sums of a finite number (r) of terms in the two series

$$1 + (m+1)x + (2m+1)x^2 + \ldots + (mr - m + 1)x^{r-1}$$

and $$1 + (m+1)^2x + (2m+1)^2x^2 + \ldots + (mr - m + 1)^2x^{r-1},$$

m being any positive integer, are respectively

$$[1 + (m-1)x - (mr+1)x^r + (mr - m + 1)x^{r+1}] \div (1-x)^2$$

and $[1 + (m^2 + 2m - 2)x + (m-1)^2x^2 - (mr+1)^2x^r$
$+ (2m^2r^2 - 2m^2r + 4mr - m^2 - 2m + 2)x^{r+1} - (mr - m + 1)^2x^{r+2}] \div (1-x)^3$.

Digitized by Google

Solution by LIONEL E. REAY, B.A.

(1) $S = 1 + (n+1)x + (2n+1)x^2 + \ldots (nr - n + 1)x^{r-1}$;

therefore $S(1-x) = 1 + n(x + x^2 + \ldots + x^{r-1}) - (nr - n + 1)x^r$,

$S(1-x)^2 = 1 + (n-1)x - (nr+1)x^r + (nr - n + 1)x^{r+1}$.

(2) $S = 1 + (n+1)^2 x + (2n+1)^2 x^2 + \ldots + (nr - n + 1)^2 x^{r-1}$,

$S(1-x) = 1 + (n^2 + 2n)x + n(3n+2)x^2 + \ldots + (2nr - 3n + 2)nx^{r-1} - (nr - n + 1)^2 x^r$,

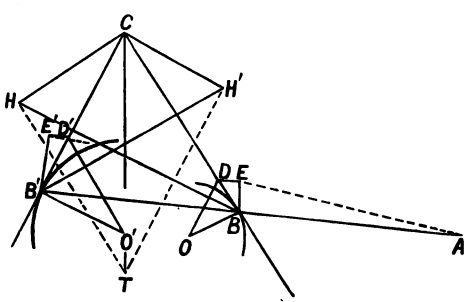
$S(1-x)^2 = 1 + (n^2 + 2n - 1)x + 2n^2x^2 + \ldots + 2n^2x^{r-1} - (n^2r^2 - 2n^2 + 2nr + 1)x^r + (nr - n + 1)^2 x^{r+1}$;

$\therefore S(1-x)^3 = 1 + (n^2 + 2n - 2)x + (n-1)^2 x^2 - (nr+1)^2 x^r + (2n^2r^2 - 2n^2r + 4nr - n^2 - 2n + 2)x^{r+1} - (nr - n + 1)^2 x^{r+2}$.

14992. (Professor NEUBERG.)—Par un point fixe A on mène une sécante quelconque qui coupe une courbe donnée Δ en B et une seconde courbe donnée Δ′ en B′. Les tangentes menées en B à Δ et en B′ à Δ′ se rencontrent en un point C. Construire la tangente en C de la courbe engendrée par ce point lorsque la sécante tourne autour de A.

Solution by the PROPOSER.

On peut résoudre le problème par la méthode de ROBERVAL. A cet effet, substituons aux courbes Δ, Δ′ leurs cercles osculateurs O, O′ aux



points B, B′. Soit BD la vitesse de B; nous la décomposons en deux autres dirigées suivant AB et une perpendiculaire à AB. Si l'on décompose de même la vitesse B′D′ de B′, les deux composantes normales BE, B′E′ dues à la rotation de BB′ autour de A sont proportionnelles à AB, AB′. Par suite, la droite AE doit passer par E′, ce qui détermine le point E′, ensuite le point D′.

Les droites O′B′, B′C ont la même vitesse angulaire; donc, si la vitesse CT du point C est décomposée en deux autres dirigées suivant CB′ et une perpendiculaire à CB′ et que CH′ représente la seconde composante, on doit avoir B′D′ : CH′ = O′B′ : B′C. Il en résulte que BH′ est perpendiculaire à O′D. Semblablement, la perpendiculaire menée de B sur

Digitized by Google

OD détermine l'extrémité de la composante CH de CT, normale à BC. Il reste à mener par H une parallèle à CB et par H′ une parallèle à CB′; ces parallèles se coupent en un point T de la tangente cherchée.

M. MANNHEIM a résolu la même question par les méthodes de sa géométrie cinématique.

12369. (Professor SYLVESTER.)—Find the polar equation (*a*) to the general involute of the logarithmic spiral, (*b*) to the *second* involute of the circle.

Solution by W. J. GREENSTREET, M.A.

The general relations $dx = r\,d\theta$, $y = r$ give for the involute of the spiral $r^n = \theta$, $nr^{n-1}dr = d\theta = dx/r$, or $ny^n dy = dx$; therefore

$$ny^{n+1} = (n+1)\,x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Hence the different orders of spirals have parabolic involutes, *e.g.*,

$n = 1$,	$r = \theta$ (ARCHIMEDES),	involute	$y^2 = 2x$;
$n = 2$,	$r^2 = \theta$ (FERMAT),	,,	$2y^3 = 3x$;
$n = \frac{1}{2}$,	$r = \theta^2$,	,,	$y^3 = 9x^2$;
$n = -2$,	$r^2\theta = 1$ (lituus),	,,	$xy = 2$;
$n = -3$,	$r^3\theta = 1$,	,,	$2xy^2 = 3$.

For $n = -1$, $r\theta = 1$ (hyperbolic), and the formula breaks down.

Here $r\,d\theta + \theta\,dr = 0$; therefore

$dx + dr/r = 0$; therefore $x + \log r = 0$, *i.e.*, $x + \log y = 0$, or $y = e^x$; therefore the involute of the hyperbolic spiral is a logarithmic spiral. For the logarithmic spiral $r = e^{m\theta}$, we have

$$dr = me^{m\theta}\,d\theta = mr\,d\theta = m\,dx; \text{ therefore } dy = m\,dx,\ y = mx + c,$$

i.e., the involute is a straight line.

For the circle $r = \cos\theta$, we have $dr = -\sin\theta\,d\theta$; therefore $y = r$, $x = -\sin\theta$; therefore $x^2 + y^2 = 1$, and involute is a circle radius unity. But the radius of $r = \cos\theta$, *i.e.*, of $(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$, is $\frac{1}{2}$; therefore the involute is a circle of double the radius. The involute of $x^2 + y^2 = 1$ or $r^2 = 1$ is found by putting $\theta = 1$ in FERMAT's spiral above, and is $2y^3 = 3x$, or in polar coordinates $2r^2 = 3\cos\theta/\sin^3\theta$.

14982. (Professor UMES CHANDRA GHOSH.)—Prove that

$$\int_0^\infty \frac{\sin(\frac{1}{4}\pi + bx)}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{b}}.$$

Solutions (I.) *by* R. CHARTRES; (II.) *by* A. M. NESBITT, M.A., *and* JOHN PRESCOTT, B.A.; (III.) *by the* PROPOSER.

(I.) $$\int_0^\infty \sin c^2x^2\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}\,c} = \int_0^\infty \cos c^2x^2\,dx$$

(TODHUNTER, *Integral Calculus*, p. 283);

Digitized by Google

therefore $$\int_0^\infty \sin(\tfrac{1}{4}\pi + c^2x^2)\,dx = \frac{\sqrt{\pi}}{2c}.$$

Let $c^2 = b$, and $x^2 = y$; therefore

$$\int_0^\infty \frac{\sin(\frac{1}{4}\pi + by)}{y^{\frac{1}{2}}}\,dy = \sqrt{\frac{\pi}{b}}.$$

(II.) Put $y = bx$; then

$$\int_0^\infty \sqrt{\frac{b}{x}}\sin(\tfrac{1}{4}\pi + bx)\,dx = \int_0^\infty \frac{\sin(\frac{1}{4}\pi + y)}{\sqrt{y}}\,dy = \frac{1}{\sqrt{2}}\int_0^\infty \left(\frac{\sin y}{\sqrt{y}} + \frac{\cos y}{\sqrt{y}}\right)dy$$
$$= 1/\sqrt{2}\,(\sqrt{\tfrac{1}{2}\pi} + \sqrt{\tfrac{1}{2}\pi}) \qquad \text{(Todhunter, p. 283)}$$
$$= \sqrt{\pi};$$

therefore $$\int_0^\infty \frac{\sin(\frac{1}{4}\pi + bx)}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{b}}.$$

(III.) $$\int_0^\infty \frac{\sin bx}{x^n}\,dx = \frac{b^{n-1}}{\Gamma(n)}\,\frac{\pi}{2\sin\frac{1}{2}n\pi} \text{ and } \int_0^\infty \frac{\cos bx}{x^n}\,dx = \frac{b^{n-1}}{\Gamma(n)}\,\frac{\pi}{2\cos\frac{1}{2}n\pi}$$

(Williamson, *Integral Calculus*, p. 167);

therefore $$\int_0^\infty \frac{\sin bx + \cos bx}{x^n}\,dx = \frac{b^{n-1}}{\Gamma(n)}\,\tfrac{1}{2}\pi\,\frac{\cos\frac{1}{2}n\pi + \sin\frac{1}{2}n\pi}{\sin\frac{1}{2}n\pi\cos\frac{1}{2}n\pi},$$

or $$\int_0^\infty \frac{\sqrt{2}\sin(\frac{1}{4}\pi + bx)}{x^n}\,dx = \frac{b^{n-1}}{\Gamma(n)}\,\tfrac{1}{2}\pi\,\frac{\sqrt{2}\sin(\frac{1}{4}\pi + \frac{1}{2}n\pi)}{\sin\frac{1}{2}n\pi\cos\frac{1}{2}n\pi};$$

therefore $$\int_0^\infty \frac{\sin(\frac{1}{4}\pi + bx)}{x^n}\,dx = \frac{b^{n-1}}{\Gamma(n)}\,\tfrac{1}{2}\pi\,\frac{\sin(\frac{1}{4}\pi + \frac{1}{2}n\pi)}{\sin\frac{1}{2}n\pi\cos\frac{1}{2}n\pi}.$$

Put $n = \frac{1}{2}$. Then

$$\int_0^\infty \frac{\sin(\frac{1}{4}\pi + bx)}{\sqrt{x}} = \frac{b^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}\,\tfrac{1}{2}\pi\,\frac{1}{\frac{1}{2}} = \frac{1}{\sqrt{b}}\,\frac{1}{\sqrt{\pi}}\,\tfrac{1}{2}\pi\,.\,2 = \sqrt{\frac{\pi}{b}}.$$

9290. (A. E. Jolliffe.)—Find the sum to n terms and also to $a+1$ terms of

$$1 - a^x + \frac{a^x(a^x-1^x)}{1^x\,.\,2^x} - \frac{a^x(a^x-1^x)(a^x-2^x)}{1^x\,.\,2^x\;3^x} + \ldots,$$

and deduce the value of

$$1 - \frac{a^x - r^x}{(r+1)^x} + \frac{(a^x-r^x)\{[a^x-(r+1)^x]\}}{(r+1)^x(r+2)^x} - \ldots$$

to $(a-r+1)$ terms, where a is not less than r, a, r, x being integers.

Solution by A. W. Poole.

Let $$S = 1 - a^x + \frac{a^x(a^x-1^x)}{1^x\,.\,2^x} - \frac{a^x(a^x-1^x)(a^x-2^x)}{1^x\,.\,2^x\,.\,3^x} + \ldots.$$

Then $$S_1 = 1, \quad S_2 = 1 - a^x,$$

$$S_3 = (1^x - a^x)\left[1 - \frac{a^x}{2^x}\right] = \frac{(1^x-a^x)(2^x-a^x)}{2^x}.$$

Digitized by Google

Writing $$S_2 = \frac{(1^x - a^x)(2^x - a^x)}{1^x \cdot 2^x},$$

we see that $$S_n = \frac{(1^x - a^x)(2^x - a^x) \dots [(n-1)^x - a^x]}{1^x \cdot 2^x \dots (n-1)^x};$$

therefore $S_{a+1} = 0$, the last term in number being $a^x - a^x$.

Let now $$\Sigma = 1 - \frac{a^x - r^x}{(r+1)^x} + \frac{(a^x - r^x)[a^x - (r+1)^x]}{(r+1)^x (r+2)^x} - \dots;$$

then $$\frac{a^x (a^x - 1^x) \dots [a^x - (r-1)^x]}{1^x \cdot 2^x \dots r^x} \Sigma_{a-r+1}$$

$$= \frac{a^x (a^x - 1^x) \dots [a^x - (r-1)^x]}{1^x \dots r^x} - \frac{a^x \dots (a^x - r^x)}{1^x \dots (r+1)^x} + \dots = (-1)^r [S_{a+1} - S_r]$$

$$= -(-1)^r S_r = (-1)^{r+1} \frac{(1^x - a^x) \dots [(r-1)^x - a^x]}{1^x \cdot 2^x \dots (r-1)^x}$$

$$= \frac{(a^x - 1^x) \dots [a^x - (r-1)^x]}{1^x \cdot 2^x \dots (r-1)^x};$$

therefore $$\Sigma_{a-r+1} = (r/a)^x.$$

14671. (R. Knowles.)—If A_r is the coefficient of x^r in the expansion of $(1-x)^{-n}$, prove that the sum of the series $1 + A_1^2 x + A_2^2 x^2 + A_3^2 x^3 + \dots$ ad inf. is a function of the form

$$(1 + p_1^2 x + p_2^2 x^2 + \dots + p_{n-2}^2 x^{n-2} + x^{n-1})/(1-x)^{2n-1},$$

in which p_r is the coefficient of x^r in the expansion of $(1+x)^{n-1}$, *e.g.*, if $n = 5$, the sum of the series

$$1 + 5^2 x + 15^2 x^2 + 35^2 x^3 + \&c. = (1 + 16x + 36x^2 + 16x^3 + x^4)/(1-x)^9.$$

Solution by Professor S. Sircom.

Let B_r be the coefficient of x^r in the expansion of $1/(1-x)_{n+1}$; then $nB_r = (n+r) A_r$; q_r the coefficient of x^r in that of $(1-x)^n$; then

$$(n-r+1) q_{r-1} = r q_r \quad \text{and} \quad (n-r) q_r = n p_r.$$

Then $1 + B_1^2 x + B_2^2 x^2 + \dots$

$= 1/n^2 (x\, d/dx + n)^2 (1 + p_1^2 x + p_2^2 x^2 + \dots)/(1-x)^{2n-1}$

$= 1/n^2 (x\, d/dx + n) [(2n-1) x (1 + p_1^2 x + p_2^2 x^2 + \dots)$

$\qquad + (1-x)(x\, d/dx + n)(1 + p_1^2 x + p_2^2 x^2 + \dots)]/(1-x)^{2n}$

$= 1/n^2 (x\, d/dx + n) \{n + [n - 1 + (n+1) p_1^2] x$

$\qquad + [(n-2) p_1^2 + (n+2) p_2^2] x^2 + \dots\}/(1-x)^{2n},$

which becomes

$$1/n^2 (x\, d/dx + n) [n + (n-1) q_1^2 x + (n-2) q_2^2 x^2 + (n-3) q_3^2 x^2 + \dots]/(1-x)^{2n},$$

in virtue of the relations between q_{r-1}, q_r, and p_r. The rest of the operation gives $(1 + q_1^2 x + q_2^2 x^2 + \dots)/(1-x)^{2n+1}$, and the theorem is proved by induction.

Digitized by Google

Geometrical Note.

By R. F. Davis, M.A.

Euclid's Propositions II. 11 and IV. 10 might very well be dispensed with if the inscription of the regular decagon in a given circle were allowed to precede that of the pentagon.

(i.) Let O be the centre and OA the radius of a given circle. Along the tangent at A take TA = 2OA. Draw TPOQ. Then shall TP be the length of the side of the regular decagon inscribed in the circle whose radius = QP (= 2OA).

For $TP.TQ = TA^2 = QP^2$; then compare (ii.).

(ii.) Let PR be the side of a regular decagon inscribed in a circle centre Q. Produce QP to T so that TP = PR, and join TR. Then $PQR = 2\pi/10 = \pi/5$; QPR (= QRP) $= 2\pi/5$, since QP = QR; $PTR = PRT = \pi/5$, since TP = PR; therefore TR = QR = QP, TR touches the circle about QPR, and $TP.TQ = TR^2 = QP^2$.

15088. (Rev. T. Wiggins, B.A.)—(1) Of numbers between 1 and 20, how many sets of 3, 4, or 5 can be found such that the square of one number in each set equals the sum of the squares of the remaining numbers in the same set? Also, if the numbers from 1 to 100 are taken, how many sets of 3 numbers satisfy the above conditions? No set should be a multiple of any other. (2) Find in *two* ways $a_1, a_2, \ldots a_9$, such that $a_1^2 = a_2^2 + a_3^2 = a_4^2 + a_5^2 = a_6^2 + a_7^2 = a_8^2 + a_9^2$, where *each* of the a's lies between 1 and 100, but the suffixes have no reference to order of magnitude.

Solution by Lt.-Col. Allan Cunningham, R.E.

(1) Assuming that in the solutions of

(i.) $c^2 = a^2 + b^2$, (ii.) $d^2 = a^2 + b^2 + c^2$, (iii.) $e^2 = a^2 + b^2 + c^2 + d^2$,

no two of the numbers are to be alike in any one solution, the solutions with numbers $\not> 20$ are as follow:

(i.) $5^2 = 3^2 + 4^2$, $13^2 = 12^2 + 5^2$, $17^2 = 15^2 + 8^2$; *total* 3.

(ii.) Writing for shortness $d\,(a, b, c)$ for $d^2 = a^2 + b^2 + c^2$,

7 (6, 3, 2); 9 (8, 4, 1); 11 (9, 6, 2); 13 (12, 4, 3); 15 (14, 5, 2);
15 (11, 10, 2); 17 (12, 9, 8); 19 (18, 6, 1); 19 (15, 10, 6); *total* 9.

(iii.) Writing for shortness $e\,(a, b, c, d)$ for $e^2 = a^2 + b^2 + c^2 + d^2$,

11 (10, 4, 2, 1); 13 (10, 8, 2, 1); 13 (10, 7, 4, 2); 15 (14, 4, 3, 2);
15 (12, 8, 4, 1); 15 (10, 8, 6, 5); 15 (11, 8, 6, 2); 15 (13, 6, 4, 2);
17 (14, 8, 5, 2); 17 (12, 10, 6, 3); 17 (13, 10, 4, 2); 17 (11, 10, 8, 2);
19 (16, 8, 5, 4); 19 (16, 10, 2, 1); 19 (14, 10, 7, 4); 19 (14, 10, 8, 1);
19 (12, 10, 9, 6); *total* 17.

Next, with numbers $\not> 100$ the solutions of (i.) are (the value of c is given for shortness)

c = 5, 13, 17, 25, 29, 37, 41, 53, 61, 65, 65, 73, 85, 85, 89, 97; *total* 16;

Digitized by Google

as c may be any *prime* of form $(4w+1)$, or any *power* or *product* of *such primes*, the products 65, 85 are counted *twice* (each giving *two* solutions).

(2) The only solutions with numbers $\not> 100$ are

$$65^2 = 39^2+52^2 = 25^2+60^2 = 63^2+16^2 = 33^2+56^2;$$
$$85^2 = 51^2+68^2 = 75^2+40^2 = 13^2+84^2 = 77^2+36^2;$$

as 65, 85 are the only numbers (<100) which contain two unequal prime factors each of form $(4w+1)$.

[*Note by the* Proposer:—In the form $\epsilon(a, b, c, d)$ 5 other sets may be found, viz.,

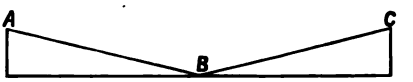
9 (2, 4, 5, 6); 14 (1, 5, 7, 11); 18 (1, 3, 5, 17); 18 (3, 5, 11, 13); 18 (5, 7, 9, 13).

Also if *two* of the numbers may be alike in the form $d(a, b, c)$, the total becomes 14; and, if *two* or *three* of the numbers in the form $e(a, b, c, d)$ are alike, the total is increased to 52.]

15107. (I. Arnold.)—There are two opposite inclined planes each inclined to the horizon at an angle of 15°, viz., AB and BC; and touching each other at B. The planes are 100 feet long each; and a small perfectly smooth metallic ball rolls down AB and then ascends BC. In what time will the ball come to rest between the two planes?

Solution by M. Peiris.

Let h be the height of either plane. When the ball comes to B it has a



velocity of $\sqrt{(2gh)}$ in the direction AB acquired in time $\sqrt{(2gh)}/g \sin 15°$. The component, $\sqrt{(2gh)}\cos 30°$, of this velocity along BC takes the ball up that plane to a height of $h\cos^2 30°$ feet in time $[\sqrt{(2gh)}/g \sin 15°]\cos 30°$, and when the ball comes down the plane CB to B it has a velocity of $\sqrt{(2gh)}\cos 30°$ in time $[\sqrt{(2gh)}/g \sin 15°]\cos 30°$. As before, the component, $\sqrt{(2gh)}\cos^2 30°$, of this velocity along AB takes the ball up that plane to a height of $h\cos^4 30°$ in time $[\sqrt{(2gh)}/g \sin 15°]\cos^3 30°$. In coming down AB the ball takes the same time. So the whole time (in seconds) taken before the ball comes to rest is the sum of the infinite series $[\sqrt{(2gh)}/g \sin 15°] + 2[\sqrt{(2gh)}/g \sin 15°]\cos 30° + 2[\sqrt{(2gh)}/g \sin 15°]\cos^2 30° + \ldots$.

This is $= [\sqrt{(2gh)}/g \sin 15°] + 2[\sqrt{(2gh)}/g \sin 15°]\cos 30°[1 + \cos 30° + \cos^2 30° + \ldots]$

$= [\sqrt{(2gh)}/g \sin 15°](1 + 2\cos 30°)[1/(1-\cos 30°)]$

$= \sqrt{(200/g \sin 15°)}[2(1+\sqrt{3})]/(2-\sqrt{3})$

$= 100$ secs. nearly,

since $h = 100 \sin 15° = 5\sqrt{2}(1+\sqrt{3})^{\frac{1}{2}}(2+\sqrt{3})$.

Digitized by Google

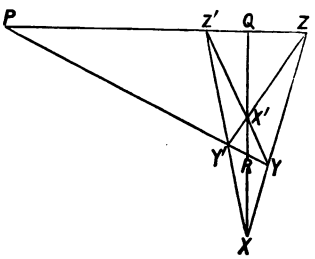
15054. (G. F. S. Hills.)—There are six confocals to $x^2/a^2+y^2/b^2=1$, such that quadrilaterals circumscribed to this ellipse are inscribed in the confocal, and the six values of λ $[x^2/(a^2+\lambda)+y^2/(b^2+\lambda)=1]$ are given by $$(\lambda^2-a^2b^2)(\lambda^2+2b^2\lambda+a^2b^2)(\lambda^2+2a^2\lambda+a^2b^2)=0.$$

Solutions (I.) *by* R. F. Whitehead; (II.) *by the* Proposer.

(I.) If the lines XY, YX′, X′Y′, Y′X be $(lx\pm my\pm nz=0)$ referred to the triangle PQR, any conic touching them is of the form

$$ax^2+by^2+cz^2=0,$$

where $l^2/a+m^2/b+n^2/c=0$.....(1).



Also any conic through XYX′Y′ will be of the form

$$(lx+my+nz)(lx+my-nz)+\theta(lx-my+nz)(-lx+my+nz)=0$$

or $$l^2x^2+m^2y^2-n^2z^2+2\mu lmxy=0$$

$[\text{where } \mu=(1+\theta)/(1-\theta)]$.

Hence $\Delta=abc$, $\Theta=bcl^2+cam^2-abn^2=-2abn^2$ [from (1)],

$$\Theta'=-am^2n^2-bn^2l^2+cl^2m^2(1-\mu^2)=abn^4/c+cl^2m^2(1-\mu^2)\text{ [from (1)]},$$

$$\Delta'=-l^2m^2n^2(1+\mu^2);$$

therefore $$\Theta'=a^2b^2n^4/\Delta-c\Delta'/n^2=\Theta^2/4\Delta+2\Delta\Delta'/\Theta.$$

This is the invariant condition that a quadrilateral may be drawn about S and in S′.

If now S be $bx^2+ay^2-ab=0$, and S′ be a confocal

$$(b+\lambda)x^2+(a+\lambda)y^2-(a+\lambda)(b+\lambda)=0,$$

we have $$\Delta=-a^2b^2,\quad \Delta'=-(a+\lambda)^2(b+\lambda)^2,$$

$$\Theta'=-(a+\lambda)(b+\lambda)[(a+b)\lambda+3ab],\quad \Theta=-ab[\lambda^2+2(a+b)\lambda+3ab];$$

whence, on reduction, we have

$$\lambda^6+2(a+b)\lambda^5+5ab\lambda^4-5a^2b^2\lambda^2-2a^2b^2(a+b)\lambda-a^3b^3=0,$$

or $$(\lambda^2-ab)(\lambda^2+2a\lambda+ab)(\lambda^2+2b\lambda+ab)=0.$$

(II.) Take the conics

$$S\equiv x^2/p^2+y^2/q^2=1,\quad S'=ax^2+2bxy+\ldots+c=0.$$

Take points whose eccentric angles are α, β, γ, δ on the first; then the pole of $\beta\gamma$ for it lies on S′ if

$$(ap^2-bq^2+c)\cos\beta\cos\gamma+(bq^2-ap^2+c)\sin\beta\sin\gamma+ap^2+bq^2+c$$
$$+2gp(\cos\beta+\cos\gamma)+2fq(\sin\beta+\sin\gamma)+2hpq\sin(\beta+\gamma)=0\ldots(\text{i.}),$$

on slight reduction.

Hence, if we draw a quadrilateral circumscribing S, touching at points α, β, γ, δ, the corners all lie on S′, if the four equations of type (i.) between $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, $\delta\alpha$ hold. This is a poristic system, and the condition for its existence is

$$\begin{vmatrix} ap^2-bq^2+c & 2hpq & 2gp \\ 2hpq & bq^2-ap^2+c & 2fq \\ 2gp & 2fq & ap^2+bq^2+c \end{vmatrix}=0.$$

Digitized by Google

In the case of confocals

$$f = g = h = 0, \quad c = -1, \quad a = 1/(p^2+\lambda), \quad b = 1/(q^2+\lambda);$$

hence $[p^2/(p^2+\lambda) - q^2/(q^2+\lambda) - 1][q^2/(q^2+\lambda) - p^2/(p^2+\lambda) - 1]$
$\times [p^2/(p^2+\lambda) + q^2/(q^2+\lambda) - 1] = 0,$

or $$(\lambda - p^2q^2)(\lambda^2 + 2q^2\lambda + p^2q^2)(\lambda^2 + 2p^2\lambda + p^2q^2) = 0.$$

Note the above condition for porism can be obtained thus: Let

$$f(\alpha\beta) = a\cos\alpha\cos\beta + b\sin\alpha\sin\beta + c + h\sin(\alpha+\beta) + g(\cos\alpha + \cos\beta) + f(\sin\alpha + \sin\beta).$$

Then the four equations $f(\alpha\beta) = 0$, $f(\beta\gamma) = 0$, $f(\gamma\delta) = 0$, $f(\delta\alpha) = 0$ exist only if a certain relation hold.

Now $f(\beta\theta) = 0$, $f(\delta\theta) = 0$ have the same roots for θ, viz. $\theta = \alpha$, $\theta = \gamma$; hence, equating the coefficients of $\cos\theta$, $\sin\theta$, and the constant term,

$$\frac{a\cos\beta + h\sin\beta + g}{a\cos\delta + h\sin\delta + g} = \frac{h\cos\beta + b\sin\beta + f}{h\cos\delta + b\sin\delta + f} = \frac{g\cos\beta + f\sin\beta + c}{g\cos\delta + f\sin\delta + c} = \lambda.$$

Eliminate $\cos\beta - \lambda\cos\delta$, $\sin\beta - \lambda\sin\delta$, $1 - \lambda$; we get

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

14877. (Robert W. D. Christie.)—Factorise

$$3e^4 + \{f(e-f)\}^2.$$

E.g.: $1483181587 = 24841 \times 59707.$

Solutions (I.) *by* H. W. Curjel, M.A., *and others*; (II.) *by* Lt.-Col. Allan Cunningham, R.E.; (III.) *by the* Proposer.

(I.) $$3e^4 + \{f(e-f)\}^2 = (3e^2 - 3ef + f^2)(e^2 + ef + f^2);$$

and, putting $e = 149$ and $f = 16$, we get

$$1483181587 = 24841 \times 59707.$$

(II.) Let $$N = \{f(e-f)\}^2 + 3e^4 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Here $$N = A^2 + 3B^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

Hence every factor (F) of N is of same form (2).

Let $$N = F'.F'' = (A' + 3B'^2).(A''^2 + 3B''^2) \quad \ldots\ldots\ldots\ldots \quad (3).$$

Then $$N = (A'A'' \mp 3B'B'')^2 + 3(A'B'' \pm B'A'')^2 \quad \ldots\ldots\ldots\ldots \quad (4).$$

Here *three* cases arise according as e, f are *odd* or *even*.

Cases (i.), (ii.) (e *odd*).—Taking the two *upper* signs in (4),

$$N = (A'A'' - 3B'B'')^2 + 3(A'B'' + B'A'')^2 = A^2 + 3B^2.$$

Now take $A' = e + \frac{1}{2}f'$, $B' = \frac{1}{2}f'$, $A'' = \frac{1}{2}f'$, $B'' = e - \frac{1}{2}f'$.

Therefore $$A = (e + \tfrac{1}{2}f')\tfrac{1}{2}f' - \tfrac{1}{2}f'(e - \tfrac{1}{2}f') = -f'(e - f');$$
$$B = (e + \tfrac{1}{2}f')(e - \tfrac{1}{2}f') + (\tfrac{1}{2}f')^2 = e^2.$$

Case (i.) (e *odd*, f *even*).—Take $f' = f$; then

$$A = -f(e-f), \quad B = e^2.$$

Digitized by Google

Case (ii.) (*e odd, f even*).—Take $f' = (e-f)$; then

$$A = -(e-f)f, \quad B = e^2.$$

Thus, in both cases, (i.), (ii.), the product (3) reduces to the required form (1).

Case (iii.) (*e even, f odd*).—Taking the two *lower* signs in (4),

$$N = (A'A'' + 3B'B'')^2 + 3(A'B'' - B'A'')^2 = A^2 + 3B^2.$$

Now take $A' = f - \frac{1}{2}e$, $B' = \frac{1}{2}e$, $A'' = f + \frac{1}{2}e$, $B'' = \frac{1}{2}e$.

Therefore $$A = (f - \tfrac{1}{2}e)(f + \tfrac{1}{2}e) + 3(\tfrac{1}{2}e)^2 = f(e-f);$$

$$B = (f - \tfrac{1}{2}e)\tfrac{1}{2}e - \tfrac{1}{2}e(f + \tfrac{1}{2}e) = -e^2;$$

so that here also the product (4) reduces to the required form (1).

Ex. of (i.).—$N = 1483181587 = 2128^2 + 3.149^4$; $e = 149$, $f(e-f) = 16.133$ take $f = 16 = f'$. Here

$$A' = 157, \quad B' = 8, \quad A'' = 8, \quad B'' = 141;$$

$$F' = 157^2 + 3.8^2 = 24841, \quad F'' = 8^2 + 3.141^2 = 59707.$$

Ex. of (ii.).—[Same as in (i.)]; $e = 149$, $f(e-f) = 16.133$; take $f = 133$, $f' = 16$; and the rest of the work is as in Case (i.).

Ex. of (iii.).—$N = 44161 = 119^2 + 3.10^4$; $e = 10$, $f = 23$. Here

$$A' = 8, \quad B' = 5, \quad A'' = 28, \quad B'' = 5;$$

$$F' = 8^2 + 3.5^2 = 139, \quad F'' = 28^2 + 3.5^2 = 859.$$

(III.) The given expression $= \{e^2 + ef + f^2\}\{e^2 + e(e-f) + (e-f)^2\}$.

If e be composite, we can eliminate f^4 and obtain

$$3g^4 + (g-1)^2 = 3(k^2+1)^4 + k^4$$

$$= \{(k^2+1)^2 + (k^2+1) + 1\}\{(k^2+1)^2 + (k^2+1)k^2 + k^4\},$$

an expression which leads to further developments.

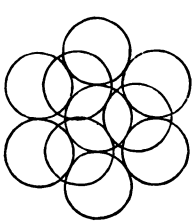
Let $k = \cot A$ or $\tan A$. Then

$$3(k^2+1)^4 + k^4 \equiv \{(k^2+1)^2 + (k^2+1) + 1\}\{(k^2+1)^2 + (k^2+1)k^2 + k^4\}$$

becomes $$3(\operatorname{cosec}^2)^4 + \cot^4 \equiv \ldots \equiv 3(\sec^2)^4 + \tan^4.$$

10236. (Artemas Martin, LL.D.)—What is the greatest number of equal spheres, each 1 inch in diameter, that can be packed in a hollow sphere whose internal diameter is 3 inches?

Solution by J. Blaikie, M.A.



It seems clear that 13 is the greatest number. Seven equal spheres may be arranged with their centres in one plane through the centre of the hollow sphere, so that the central sphere is touched by the other six. Three spheres may be placed above and three below, and thus the central sphere will be touched by twelve spheres, and each of the enveloping spheres will touch five equal spheres as well as the outer sphere of 3 inches diameter. It would also be possible to arrange the twelve enveloping spheres at the corners of an icosahedron, and in that case each would touch the central

Digitized by Google

sphere, but the enveloping spheres would not touch each other. For, if O, A, B, C, D, E be six adjacent vertices of an icosahedron of which A, B, C, D, E are in one plane, and if OL be drawn perpendicular to this plane, it is easy to calculate that

$$OL^2 = AB^2(3+\sqrt{5})/(10+4\sqrt{5}),$$

which is greater than $\frac{1}{4}AB$, since $4(3+\sqrt{5})$ is greater than $10+4\sqrt{5}$. Thus, if OL be produced its own length to O′, an equal sphere whose centre was at O′ would not touch the sphere O, and the central sphere touching the twelve spheres whose centres are at the vertices of the icosahedron and whose radii are each equal to half an edge, is less than each of the enveloping spheres. If then an equal sphere be placed at the centre, each of the enveloping spheres will be pushed out radially, and they will cease to touch each other, but the additional space is clearly insufficient to admit a fourteenth sphere.

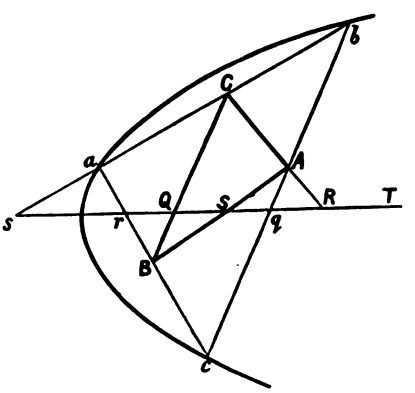
14961. (H. Bateman.)—The axis of a parabola circumscribing a triangle is a Simson line with respect to the triangle formed by joining the middle points of the sides.

Solutions (I.) *by* Professor Sanjána, M.A.; (II.) *by* A. M. Nesbitt, M.A., *and* L. Isserlis.

(I.) Let a parabola be circumscribed about the triangle *abc*, so that *qrs*, the axis, makes an angle θ with *bc*; let ABC be the triangle formed by joining the mid-points of the sides of *abc*. Then

$$\angle CQR = \theta,\ AST = \theta - B,$$
$$CRT = \theta + C;$$

and it is well known that the distances of B and C from that Simson-line of ABC which makes the angle θ with BC are respectively



$$2R\sin\theta\sin(\theta-B)\cos(\theta+C) \quad \text{and} \quad 2R\sin\theta\sin(\theta+C)\cos(\theta-B),$$

where R is the radius of the circumcircle of ABC.

If the coordinates of *a*, *b*, *c* be x_1y_1, x_2y_2, x_3y_3 respectively, we have

$$y_2^2 - y_1^2 = 4a(x_2-x_1);$$

therefore $$(y_2+y_1)(y_2-y_1)/(x_2-x_1) = 4a;$$

therefore $$2y_c\tan(\theta-B) = 4a \quad \text{or} \quad y_c = 2a\cot(\theta-B),$$

Digitized by Google

y_c being the ordinate of C. Similarly,

$$y_b = 2a \cot(\theta + C), \quad y_a = 2a \cot \theta.$$

But $y_c - y_a = CA \sin(\theta + C) = 2R \sin B \sin(\theta + C)$;

so that $2a\{\cot(\theta - B) - \cot\theta\} = 2R \sin B \sin(\theta + C)$

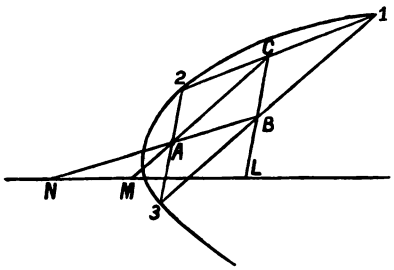
and $a = R \sin\theta \sin(\theta - B) \sin(\theta + C)$.

Thus $y_b = 2R \sin\theta \sin(\theta - B) \cos(\theta + C)$

and $y_c = 2R \sin\theta \sin(\theta + C) \cos(\theta - B)$.

Hence the axis qrs is a SIMSON-line of the triangle ABC.

(II.) Let 1 2 3 be the triangle, A, B, C the mid-points of the sides, and let BC meet the axis in L, CA in M, and AB in N. Then, if the coordinates of 1 be $(am_1^2, 2am_1)$, &c., the equation to AC (which bisects 2 3 and is parallel to 1 3) is $\frac{y - a(m_2 + m_3)}{x - \frac{1}{2}a(m_2^2 + m_3^2)} = \frac{2}{m_1 + m_3}$; therefore the abscissa of M is $\frac{1}{2}a(m_2^2 - \Sigma)$, where



$$\Sigma \equiv m_2 m_3 + m_3 m_1 + m_1 m_2,$$

and the equation of a line through M perpendicular to AC is

$$y/[x - \tfrac{1}{2}a(m_2^2 - \Sigma)] = -\tfrac{1}{2}(m_1 + m_3).$$

Interchanging m_1 and m_2, we get the line through L perpendicular to BC, and at once we find that these meet in the point $(-a\Sigma, \frac{1}{4}a\Pi)$, where

$$\Pi \equiv (m_2 + m_3)(m_3 + m_1)(m_1 + m_2).$$

Symmetry shows that the line through N perpendicular to AB passes through the same point. It is, moreover, easily seen that this point—call it P—is on the circle through ABC. For the axis (K) and the three sides of the triangle ABC form a system of four lines, and it is known that in such a system the four circles round the four triangles obtained by suppressing each line in turn have a common point. But the circle AMN circumscribes the triangle formed by omitting BC, the circle BNL that formed by omitting CA, and the circle CLM that formed by omitting AB; and these obviously all pass through P; therefore the circle ABC passes through P. Hence the perpendiculars from P, a point on the circle ABC, to the sides of the triangle ABC, meet those sides in three points L, M, N, which lie on the axis of the parabola.

15051. (R. KNOWLES.)—Prove that the sum to r terms of the series $1 - 6 + 15 - 28 \ldots \pm r(2r-1)$, when r is an even number, $= -\frac{1}{2}r(2r+1)$, and, when r is an odd number, $= \frac{1}{2}(r+1)(2r-1)$.

Digitized by Google

Solutions (I.) *by* Lt.-Col. ALLAN CUNNINGHAM, R.E., *and* C. A. B. GARRETT; (II.) *by* H. B. GOODWIN *and* J. H. TAYLOR, M.A.; (III.) *by* T. PEELE, A.I.A.

(I.) Let S_r = sum of r terms, and take the terms in pairs. When r is even,

$$\begin{aligned} S_r &= (1-6)+(15-28)+(45-66)+\&c. \\ &= -(5+13+21+\&c.\text{ to }\tfrac{1}{2}r\text{ terms}) \\ &= -\left\{10+(\tfrac{1}{2}r-1)\,8\right\}\tfrac{1}{2}r = -\tfrac{1}{2}r(2r+1). \end{aligned}$$

When r is odd,

$$\begin{aligned} S_r &= 1+(-6+15)+(-28+45)+\&c. \\ &= 1+9+17+25+\&c.\text{ to }\tfrac{1}{2}(r+1)\text{ terms} \\ &= \left\{2+\tfrac{1}{2}(r-1)\,8\right\}\tfrac{1}{2}(r+1) = \tfrac{1}{2}(r+1)(2r-1). \end{aligned}$$

(II.) When the number of terms is even, the series may be written

$$\begin{aligned} &2-2\,.\,2^2+2\,.\,3^2-2\,.\,4^2\ldots \\ &-1+\ \ 2\ -\ \ 3\ \ +\ \ 4\ \ \ldots\text{ to }r\text{ terms}, \end{aligned}$$

or

$$2\,(-3-7-11-15\ldots\text{ to }\tfrac{1}{2}r\text{ terms})+\tfrac{1}{2}r\,;$$

where $S = 2\,.\,\frac{1}{4}r\left\{-6+(\frac{1}{2}r-1)(-4)\right\}+\frac{1}{2}r = -\frac{1}{2}r(2r+2)+\frac{1}{2}r = -\frac{1}{2}r(2r+1)$. If the number of terms is odd, and $= r$, then $(r-1)$ is even; consequently, by the formula proved above, the sum of $(r-1)$ terms is $-\frac{1}{2}(r-1)(2r-1)$. Adding the r-th term, namely, $r(2r-1)$, we have for the whole sum

$$-\tfrac{1}{2}(r-1)(2r-1)+2r^2-r \text{ or } \tfrac{1}{2}(2r^2+r-1) = \tfrac{1}{2}(r+1)(2r-1).$$

(III.) Let t_p = numerical value of the p-th term, $s_p = t_{2p-1}-t_{2p}$, S_p = sum of p terms of the original series, $s_p = -(8p-3)$;

$$S_{2n} = \sum_{p=1}^{p=n} s_p = -n(4n+1).$$

Put $2n = r$: $\quad S_r = -\frac{1}{2}r(2r+1)$ (r even),

$$S_{2n-1} = S_{2n}+t_{2n} = -n(4n+1)+2n(4n-1) = n(4n-3).$$

Put $2n-1 = r$: $\quad S_r = \frac{1}{2}(r+1)(2r-1)$ (r odd).

15084. (A. M. NESBITT, M.A.) — If u_r denote the chance that, of r letters intended for r different recipients and placed *at random* into r envelopes correctly addressed, not one shall reach its proper destination (disregarding errors of delivery, &c.), it is fairly obvious *a priori* that

$$u_n+u_{n-1}/1!+u_{n-2}/2!+\ldots+u_0/n! = 1.$$

Solve this difference equation *directly*; *i.e.*, prove that u_n = the first $n+1$ terms of the expansion of e^{-1} without reference to the problem from which it originally sprang.

Solution by Professor NANSON.

If $v_n = n!\,u_n$ and E is defined by $\mathrm{E}v_n = v_{n+1}$, the given equation is

$$(\mathrm{E}+1)^n v_0 = n!.$$

Hence $u_n = \dfrac{1}{n!}\mathrm{E}^n v_0 = \Sigma(-1)^r\dfrac{1}{r!\,(n-r)!}(\mathrm{E}+1)^r v_0 = \Sigma(-1)^r\dfrac{1}{(n-r)!}$, the result required.

Digitized by Google

8276. (Professor CATALAN.)—Démontrer la formule

$$\tfrac{1}{4}\pi = 1-\tfrac{1}{3}+\tfrac{1}{5}-\ldots\mp\frac{1}{2n-1}-2n\int_0^{\frac{1}{2}\pi}\cos 2nx\log_e(2\cos\tfrac{1}{2}x)\,dx.$$

Solution by A. M. NESBITT, M.A.

Let $u_n = 2n\int_0^{\frac{1}{2}\pi}\cos 2nx\log(2\cos\frac{1}{2}x)\,dx$

$$= \left[\log(2\cos\tfrac{1}{2}x)\sin 2nx\right]_0^{\frac{1}{2}\pi} + \tfrac{1}{2}\int_0^{\frac{1}{2}\pi}\sin 2nx\tan\tfrac{1}{2}x\,dx$$

$$= \tfrac{1}{2}\int_0^{\frac{1}{2}\pi}\sin 2nx\frac{1-\cos x}{\sin x}\,dx;$$

therefore $u_{n-1} = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi}\sin(2n-2)x\frac{1-\cos x}{\sin x}\,dx,$

$$u_n-u_{n-1} = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi}[\sin 2nx-\sin(2n-2)x]\frac{1-\cos x}{\sin x}\,dx$$

$$= \int_0^{\frac{1}{2}\pi}(1-\cos x)\cos(2n-1)x\,dx$$

$$= \int_0^{\frac{1}{2}\pi}\{\cos(2n-1)x-\tfrac{1}{2}[\cos 2nx+\cos(2n-x)x]\}\,dx\ldots\ldots(A),$$

and this, if $n>1$, $= \dfrac{\sin(2n-1)}{2n-1}\tfrac{1}{2}\pi,$

since the other terms when integrated vanish at both limits if $n>1$;

therefore $u_n-u_{n-1} = \dfrac{\sin(2n-1)}{2n-1}\tfrac{1}{2}\pi,$

$$u_{n-1}-u_{n-2} = \frac{\sin(2n-3)}{2n-3}\tfrac{1}{2}\pi,\quad \ldots = \ldots,\quad u_2-u_1 = \frac{\sin\frac{3}{2}\pi}{3}.$$

Again, $u_1-u_0 = \int_0^{\frac{1}{2}\pi}[\cos x-\tfrac{1}{2}(\cos 2x+1)]\,dx$ [as in (A)]

$$= \left[\sin x-\tfrac{1}{4}\sin 2x-\tfrac{1}{2}x\right]_0^{\frac{1}{2}\pi} = 1-\tfrac{1}{4}\pi;$$

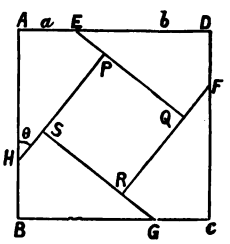
therefore, by addition,

$$u_n+\tfrac{1}{4}\pi = 1-\tfrac{1}{3}+\tfrac{1}{5}\ldots\pm 1/(2n-1).$$

13967. (R. TUCKER, M.A.) — In the figure, ABCD is a given square,

$$AE = DF = CG = BH = a,$$

and PQRS is another square. Find when PQRS is a *maximum* (θ variable).



Solution by the late Rev. T. MITCHESON, B.A.

It would appear that PQRS is a *maximum* when HPEA is a *minimum*; which is so when HP coincides with HE. Hence the value required is

$$(a+b)^2-2ab = a^2+b^2.$$

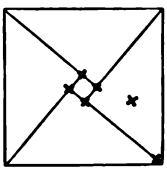
Digitized by Google

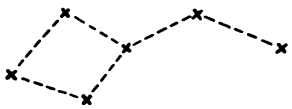
Or, thus: join HE, and let $\angle EHP$ be ϕ; then

$$\text{area of HPEA} = \tfrac{1}{2}\{ab + (a^2+b^2)\sin(\theta-\phi)\cos(\theta-\phi)\}$$
$$= \tfrac{1}{4}\{2ab + (a^2+b^2)\sin 2(\theta-\phi)\}.$$

This is obviously a minimum when $\theta = \phi$, &c.

NOTE.—The figure is a common one in bamboo chairs. For notes on this, the Bride's Chair, Mr. TUCKER refers to pp. 633, 637 of CLIFFORD's *Mathematical Papers*. He also contributes the diagrams given below,





the resemblance to the form of the constellation of Cassiopeia, or "The Lady in her Chair," having been pointed out by the late Mr. H. C. LEVANDER, M.A.

15038. (Professor N. BHATTACHARYYA.)—If Ω, Ω' be the BROCARD points of a triangle ABC, and O be the centre of the BROCARD ellipse, prove that

$$(\Omega A^2 - \Omega' A^2)/c^2 + (\Omega B^2 - \Omega' B^2)/a^2 + (\Omega C^2 - \Omega' C^2)/b^2$$
$$= (\Omega' A^2 - \Omega A^2)/b^2 + (\Omega' B^2 - \Omega B^2)/c^2 + (\Omega' C^2 - \Omega C^2)/a^2$$
$$= 2(1/b^2 + 1/c^2)(\Omega A^2 - O A^2) + 2(1/c^2 + 1/a^2)(\Omega B^2 - O B^2)$$
$$+ 2(1/a^2 + 1/b^2)(\Omega C^2 - O C^2)$$
$$= 1 - 16\Delta^2/(b^2c^2 + c^2a^2 + a^2b^2),$$

where Δ is the area of the triangle.

Solutions (I.) *by* R. TUCKER, M.A.; (II.) *by* H. L. TRACHTENBERG.

(I.) Taking the coordinates of Ω, Ω' to be $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$, we have

$$\Omega A = \gamma_1 \operatorname{cosec}\omega = b^2 c \cdot 2\Delta/\lambda^2 \quad \text{and} \quad \Omega' A = \beta_2 \operatorname{cosec}\omega = bc^2 \cdot 2\Delta/\lambda^2.$$

Hence (i.) $= \Sigma \Omega A^2/c^2 - \Sigma \Omega' A^2/c^2 = (\nu^4 - \lambda^4)(2\Delta \operatorname{cosec}\omega)^2/\lambda^4$
$= (\nu^4 - \lambda^4)/\lambda^2 = e^2$.

For (ii.) we get, in like manner, e^2. Now

$$2OA^2 = \Omega A^2 + \Omega' A^2 - (2a_1 e)^2 = (b^2+c^2)\, b^2c^2/\lambda^2 - a^2b^2c^2e^2/2\lambda^2;$$

hence $$2\Sigma (b^2+c^2)\, OA^2/b^2c^2 = 2(\nu^4 + \lambda^4)/\lambda^2 - e^2$$

and $$2\Sigma (b^2+c^2)\, \Omega A^2/b^2c^2 = 2(\nu^4 + \lambda^4)/\lambda^2;$$

hence (iii.) $= e^2$, which is (iv.); therefore, &c.

(References are to MILNE's *Companion*, pp. 107–110, or to my "Triplicate Ratio Circle," *Quart. Jour.*, Vol. XIX., No. 70, where $\lambda^2 = \Sigma a^2b^2$, $\nu^4 = \Sigma a^4$, $2a_1$ and e are the major axis and eccentricity of the BROCARD ellipse.)

Digitized by Google

(II.) $\Omega A/\sin\omega = b/\sin A$ and $\Omega' A/\sin\omega = c/\sin A$;

therefore $\Sigma(\Omega A^2 - \Omega' A^2)/c^2 = \sin^2\omega \Sigma(b^2-c^2)/c^2\sin^2 A$

$= \sin^2\omega/4\Delta^2 \Sigma(b^4 - b^2c^2)$ .....(1).

But $\Delta = \Sigma$ triangle $B\Omega C = \Sigma \frac{1}{2}(B\Omega \cdot C\Omega \sin C) = \Sigma Ca \sin^2\omega/2\sin B$

$= (\sin^2\omega/4\Delta)\Sigma c^2a^2$.

Hence $\sin^2\omega = 4\Delta^2/\Sigma c^2a^2$;

therefore $\Sigma(\Omega A^2 - \Omega' A^2)/c^2 = \Sigma(b^4-b^2c^2)/\Sigma c^2a^2 = 1-(\Sigma 2b^2c^2 - \Sigma a^4)/\Sigma c^2a^2$.

But $\Sigma 2b^2c^2 - \Sigma a^4 = (a+b+c)(b+c-a)(c+a-b)(a+b-c) = 16\Delta^2$.

Hence $\Sigma(\Omega A^2 - \Omega' A^2)/c^2 = 1 - 16\Delta^2/(b^2c^2 + c^2a^2 + a^2b^2)$.

Again, $\Sigma(\Omega' A^2 - \Omega A^2)/b^2 = \sin^2\omega \Sigma(c^2-b^2)/b^2\sin^2 A$

$= (\sin^2\omega/4\Delta^2)\Sigma(c^4 - b^2c^2) = \Sigma(\Omega A^2 - \Omega' A^2)/c^2$

by (1). Again, since Ω, Ω' are the foci of the BROCARD ellipse, O is the mid-point of $\Omega\Omega'$, the base of triangle $\Omega A\Omega'$; therefore

$$O A^2 = \tfrac{1}{2}(\Omega A^2 + \Omega' A^2) - \tfrac{1}{4}(\Omega\Omega'^2);$$

$$\therefore \Sigma 2(1/b^2 + 1/c^2)(\Omega A^2 - O A^2) = \Sigma(1/b^2 + 1/c^2)\{\Omega A^2 - \Omega' A^2 + \tfrac{1}{2}(\Omega\Omega'^2)\}$$

$$= \Sigma(1/b^2 + 1/c^2)\tfrac{1}{2}(\Omega\Omega'^2),$$

since it was proved that $\Sigma(\Omega A^2 - \Omega' A^2)/c^2 = \Sigma(\Omega' A^2 - \Omega A^2)/b^2$.

But $\Omega\Omega'^2 = 4R^2\sin^2\omega(1 - 4\sin^2\omega)$

$= a^2b^2c^2/\Sigma b^2c^2\{1 - 16\Delta^2/(b^2c^2 + c^2a^2 + a^2b^2)\}$;

therefore $\Sigma 2(1/b^2 + 1/c^2)(\Omega A^2 - O A^2)$

$= \{\Sigma(1/b^2 + 1/c^2)a^2b^2c^2/2\Sigma b^2c^2\}\{1 - 16\Delta^2/(b^2c^2 + c^2a^2 + a^2b^2)\}$

$= 1 - 16\Delta^2/(b^2c^2 + c^2a^2 + a^2b^2)$,

since $\Sigma(1/b^2 + 1/c^2)a^2b^2c^2/2\Sigma b^2c^2 = 1$.

15011. (LIONEL E. REAY, B.A.) — Given the circumcircle and orthocentre, construct the triangle, being given also (1) a point on one of the sides, (2) one angle of the triangle.

Solutions (I.) *by* ALETROP *and very many others*; (II.) *by the* PROPOSER.

(I.) (1) Soit E le point d'intersection du circumcercle et de la perpendiculaire HD menée par l'orthocentre H au côté BC, sur lequel est le point donné P. On sait que HD = DE. Donc un premier lieu du point D est la circonférence qui a pour centre le milieu de HO et dont le rayon est $\frac{1}{2}$R. D'ailleurs, étant $\angle$ HDP = 90°, un second lieu de D est la circonférence décrite sur HP comme diamètre. Le point D reste donc déterminé, comme un de deux points possibles; &c.

(2) Le circumcercle et l'angle A étant donnés, le côté a reste déterminé. Or, d'après un théorème connu, le lieu de l'orthocentre de tous les triangles inscrits dans une même circonférence et qui ont un côté commun a est une autre circonférence égal à la donnée et dont le centre est le point O′ symétrique de O par rapport au côté a; donc en faisant

Digitized by Google

centre en O′ on décrit avec le rayon R une circonférence dont les intersections avec le circumcercle sont les sommets B et C ; &c.

Remarque.—Ce second problème pourrait se résoudre aussi au moyen de la formule connue $HA = 2R\cos A$, qui permet de déterminer sur le circumcercle le sommet A.

(II.) The locus of both D, the foot of the perpendicular on, and D′, the mid-point of, the side a, is the nine-point circle, which is known.

(1) Locus of D is also the circle on HP as diameter (H orthocentre, P given point).

(2) Length of a is known; therefore OD′ is known. This gives a second locus for D′.

8091. (A. Gordon, M.A.)—If ABCD is a quadrilateral, M, N the mid-points of AC, BD, prove that (1) MN cuts AB (in P), DC (in R) in segments inversely proportional; (2) if MN cuts CB, AD in Q, S, then

$$MP : MR = MQ : MS \quad \text{and} \quad NP : NR = NS : NQ.$$

Solutions (I.) *by* C. E. Hillyer, M.A., *and others*;
(II.) *by* A. M. Nesbitt, M.A.

(I.) Since QRN is a transversal of the triangle BCD bisecting BD,

$$QC/QB = CR/RD;$$

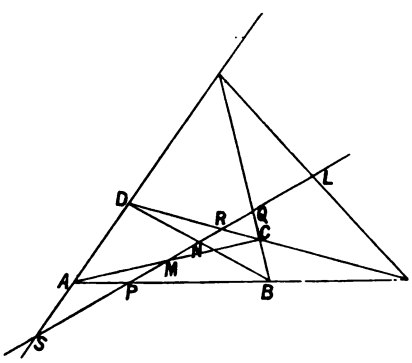
and, since QMP is a transversal of the triangle ABC bisecting AC,

$$QC/QB = AP/PB;$$

$$\therefore\ CR/RD = AP/PB;$$

and similarly

$$AP/PB = SA/SD.$$



Again, since AC is a transversal of the triangle PBQ,

$$PM/MQ = AP/AB.BC/CQ,$$

and, since AC is a transversal of the triangle DRS,

$$MR/SM = CR/CD.DA/AS;$$

but, from (1), $AP/AB = CR/CD$, and $BC/CQ = DA/AS$;

therefore $PM/MQ = MR/SM$ or $PM/MR = MQ/SM$.

Similarly, $PN/NR = SN/NQ$.

Remarks.—(i.) Since

$$PM + MR = PR = PN + NR, \quad \text{and} \quad SM + MQ = SQ = SN + NQ,$$

we have $PM/MQ = MR/SM = PN/SN = NR/NQ$,

Digitized by Google

each ratio being equal to PR/SQ. Also

$$PM/MR = (PM+MQ)/(MR+SM) = PQ/SR = MQ/SM$$

and $$PN/NR = (SN-PN)/(NQ-NR) = SP/RQ = SN/NQ.$$

(ii.) $$PM \cdot SM - PN \cdot NQ = MN^2,$$

for $$PM \cdot SM - PN \cdot NQ = PR/SQ\,(MQ \cdot SM - SN \cdot NQ)$$
$$= PR/SQ\,(MN \cdot SM + NQ \cdot SM - SM \cdot NQ - MN \cdot NQ)$$
$$= MN\,(MR/SM \cdot SM - NR/NQ \cdot NQ)$$
$$= MN\,(MR-NR) = MN^2.$$

(iii.) This can be extended to the case of a complete quadrilateral, and, if MN meet the third diagonal in L, L, M, N are the centres of the three involutions determined by the points S, P, R, Q. Moreover

$$QL \cdot SL - PN \cdot NQ = NL^2 \quad \text{and} \quad QL \cdot SL + PM \cdot SM = ML^2;$$

therefore $$PM \cdot SM + PN \cdot NQ = ML^2 - NL^2;$$

therefore, from (ii.), $PM \cdot SM = \frac{1}{2}(ML^2 - NL^2 + MN^2) = MN \cdot ML$;

so that N, L are conjugate points of the involution of which M is the centre, and P, S and R, Q pairs of conjugate points. Similarly, L, M belong to the involution of which N is the centre, and M, N to that of which L is the centre.

Apart from the quadrilateral we have the general theorem that, of the centres of the three involutions determined by four collinear points any two are conjugate points of that involution of which the third is the centre.

(II.) Let MN bisect the third diagonal EF in O. Draw $A\alpha$, $B\beta$, $C\gamma$, $D\delta$ parallel to MN. Then, clearly,

$$O\alpha = O\gamma, \quad O\beta = O\delta;$$

therefore (1)

$$O\alpha/O\beta = O\gamma/O\delta$$

or $$PA/PB = RC/RD.$$

(2) Draw MJ, ML parallel to AD and DC; NI, NK parallel to CB and CD. Then

$$E\gamma/EO = F\alpha/FO$$

or $$EC/ER = FA/FS,$$

i.e., $$2ML/ER = 2MJ/FS,$$

whence $MP/PR = MQ/QS$;

and therefore

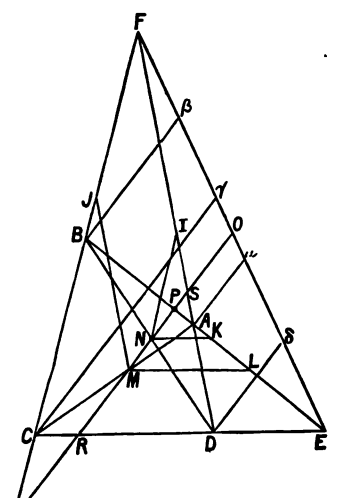
$$MP/MR = MQ/MS.$$

So also $F\beta/FO = E\delta/EO$,

$$FB/FQ = ED/ER,$$

i.e., $$2NI/FQ = 2NK/ER,$$

whence $SN/SQ = NP/NR$; and therefore $SN/NQ = PN/NR$.



Digitized by Google

15143. (J. S. Lawson, M.A.)—Sum the series

$$3-20x-30x^2-290x^3-\ldots \text{ ad inf.},$$

where $x<\frac{1}{3}$, the series being recurring.

Solution by Rev. T. Roach, M.A.

Let $1-px-qx^2$ be scale of relation.

$\therefore\ -30+20p-3q=0,\ -290+30p+20q=0\ ;\ \therefore\ p=3,\ q=10\ ;$

$\therefore\ (1-3x-10x^2)\,S=3-29x\ ;\ \therefore\ S=\dfrac{3-29x}{1-3x-10x^2}=\dfrac{5}{1+2x}-\dfrac{2}{1-5x}.$

7390. (W. S. McCay, M.A.)—If the straight lines joining the feet of perpendiculars on the sides of a triangle ABC from any point on the circumscribed circle cut the sides at angles A′, B′, C′, show that (1) its equation in areal coordinates referred to the triangle is

$$x\cot A'+y\cot B'+z\cot C'=0\ ;$$

(2) this equation is a cubic in a single parameter (either $\cot A'$, $\cot B'$, or $\cot C'$), and the envelope of the line is therefore a curve of the fourth degree with three cusps; (3) the equation of the line and its first and second differentials are all satisfied when $A'+B'+C'=(2n+1)\frac{1}{2}\pi$, and therefore the tangents at the cusps are inclined to each other at angles of 120° [the angles being measured so that $B'-C'=A+\pi$, $C'-A'=B-\pi$, $A'-B'=C-\pi$].

Solution by Professor Nanson.

Taking the point as pole and a diameter d as initial line, let the vectorial angles of A, B, C be α, β, γ; then BC, CA, AB are

$$u\equiv r\cos(\beta+\gamma-\theta)-d\cos\beta\cos\gamma=0,$$
$$v\equiv r\cos(\gamma+\alpha-\theta)-d\cos\gamma\cos\alpha=0,$$
$$w\equiv r\cos(\alpha+\beta-\theta)-d\cos\alpha\cos\beta=0\ ;$$

and the feet of the perpendiculars lie on

$$t\equiv r\cos(\alpha+\beta+\gamma-\theta)-d\cos\alpha\cos\beta\cos\gamma=0.$$

Now we have identically

$$u\cot\alpha\sin(\beta-\gamma)+v\cot\beta\sin(\beta-\gamma)+w\cot\gamma\sin(\alpha-\beta)$$
$$=-t\operatorname{cosec}\alpha\operatorname{cosec}\beta\operatorname{cosec}\gamma\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta),$$

and, if x, y, z are the areal coordinates of r, θ,

$$x:y:z=u\sin(\beta-\gamma):v\sin(\gamma-\alpha):w\sin(\alpha-\beta)\ ;$$

so that the areal equation of the pedal line is

$$x\cot\alpha+y\cot\beta+z\cot\gamma=0\ldots\ldots\ldots\ldots\ldots\ldots\ (1),$$

where α, β, γ are the angles it makes with the sides. Since

$$\tan(\beta-\gamma)=\tan A,\quad \tan(\gamma-\alpha)=\tan B,\quad \tan(\alpha-\beta)=\tan C,$$

Digitized by Google

(1) is cubic in a single parameter; so that the envelope is a unicursal class cubic, and is therefore of order 4 and has three cusps.

Writing λ, μ, ν for $\cot\alpha$, $\cot\beta$, $\cot\gamma$, the condition for the concurrency of (1) and its first and second derived equations is the vanishing of the determinant whose first row is λ, $1+\lambda^2$, $\lambda(1+\lambda^2)$, that is to say, it is

$$(\lambda+\mu+\nu-\lambda\mu\nu)(\mu-\nu)(\nu-\lambda)(\lambda-\mu) = 0,$$

whence for a cusp we have $\alpha+\beta+\gamma = (n+\frac{1}{2})\pi$, and hence the tangents at the cusps are inclined to each other at angles of 120°.

14812. (R. Knowles.)—If

$$\mathrm{F}_1x/(1-x)^n = 1+2^{n-1}x+3^{n-1}x^2+\ \ldots,$$

and

$$\mathrm{F}_2x/(1-x)^n = 1+3^{n-1}x+5^{n-1}x^2+\ \ldots,$$

show that $\mathrm{F}_1x.\mathrm{F}_2x$ is a symmetrical function of the degree $(2n-3)$ in x, the sum of whose coefficients $= \{(n-1)(n-2)\ldots 2\}^2.2^{n-1}$, *e.g.*, if $n=4$,

$$\mathrm{F}_1x.\mathrm{F}_2x = (1+27x+116x^2+116x^3+27x^4+x^5)/(1-x)^8.$$

Solution by Professor S. Sircom.

If $\mathrm{F}_1x = 1+p_1x+p_2x^2+\ldots+p_1x^{n-3}+x^{n-2}$, a symmetrical function of the degree $(n-2)$ in x, then

$$(x\,d/dx+1)\,\mathrm{F}_1x/(1-x)^n = 1+2^nx+3^nx^2+\ldots$$
$$= [nx\mathrm{F}_1x+(1-x)(x\,d/dx+1)\,\mathrm{F}_1x]/(1-x)^{n+1}.$$

In the numerator put $x=1$; then the sum of the new coefficients is

$$n(1+p_1+p_2+\ldots);\quad \text{hence}\quad 1+p_1+p_2+\ldots = (n-1)!.$$

Performing the operations, the numerator becomes

$$1+(2p_1+n-1)x+[3p_2+(n-2)p_1]x^2+\ldots+(2p_1+n-1)x^{n-2}+x^{n-1},$$

a symmetrical function of the order $(n-1)$ in x.

Similarly, if $\mathrm{F}_2x = 1+q_1x+q_2x^2+\ldots+q_1x^{n-2}+x^{n-1}$,

then $(2x\,d/dx+1)\,\mathrm{F}_2x/(1-x)^n = 1+3^nx+5^nx^2+\ldots$

$$= [2nx\mathrm{F}_2x+(1-x)(2x\,d/dx+1)\,\mathrm{F}_2x]/(1-x)^{n+1}.$$

Putting $x=1$ as before, the sum of the coefficients is $2n(1+q_1+q_2+\ldots)$; hence, by induction, $1+q_1+q_2+\ldots = (n-1)!\,2^{n-1}$.

Performing the operations, the numerator becomes

$$1+(3q_1+2n-1)x-[5q_2+(2n-3)q_1]x^2+\ldots+(3q_1+2n-1)x^{n-1}+x^n,$$

a symmetrical function of the degree n in x.

Hence the theorem by induction.

15128. (Robert W. D. Christie.)—Let a, b be any two consecutive members of the series (continuant) 1, 2, 3, 5, 8, 13, &c. Then

$$4a^2b-4ab^2+1 = a_n^2.$$

It is required to find the series a_n.

Digitized by Google

Solutions (I.) *by* J. Blaikie, M.A., *and* Lt.-Col. Allan Cunningham, R.E.; (II.) *by* Professor Sircom *and the* Proposer; (III.) *by* Professor Sanjána, M.A.

(I.) Expressing the series 1, 2, 3, 5, 8, ... by $u_1, u_2, u_3, \ldots$, we have

$$\begin{aligned} a_n^2 &= 4u_n^3 u_{n-1} - 4u_n u_{n-1}^3 + 1 = 4u_n u_{n-1}(u_n^2 - u_{n-1}^2) + 1 \\ &= 4(u_n + u_{n-1}) u_n u_{n-1}(u_n - u_{n-1}) + 1 \\ &= 4u_{n+1} u_n u_{n-1} u_{n-2} + 1, \end{aligned}$$

from the law of formation; therefore

$$a_n^2 - a_{n-1}^2 = 4u_n u_{n-1} u_{n-2}(u_{n+1} - u_{n-3}).$$

It is easily found by trial that $a_n - a_{n-1} = 2u_n u_{n-2}$; therefore

$$\begin{aligned} a_n + a_{n-1} &= 2u_{n-1}(u_{n+1} - u_{n-3}) = 2u_{n+1} u_{n-1} - 2u_{n-1} u_{n-3} \\ &= a_{n+1} - a_n - (a_{n-1} - a_{n-2}), \end{aligned}$$

by previous relation; therefore

$$a_{n+1} = 2(a_n + a_{n-1}) - a_{n-2},$$

a relation which gives each term of the series 1, 5, 11, 31, 79, 209, ... after the first three.

If we prolong each series backwards, and place the terms of the second series in their proper intermediate places with regard to the terms of the first series, we obtain as the complete series

$$\begin{array}{llllllllllll} \ldots, & +5, & -3, & +2, & -1, & +1, & 0, & 1, & 1, & 2, & 3, & 5, \ldots, \\ \ldots, & -31, & -11, & -5, & -1, & -1, & 1, & 1, & 5, & 11, & 31, & \ldots. \end{array}$$

(II.) If a, b are the n-th and $(n-1)$-th members of the given series, then $a_n = 2ab + (-1)^n$. The series consists of the coefficients of

$$(1+x)/(1-x-x^2) = (1+x)/[(1-\alpha x)(1-\beta x)],$$

where $\alpha\beta = -1$, $\alpha + \beta = 1$, $(\alpha - \beta)^2 = 5$. Then

$$\begin{aligned} ab &= \tfrac{1}{5}[(\alpha+1)\alpha^{n-1} - (\beta+1)\beta^{n-1}][(\alpha+1)\alpha^{n-2} - (\beta+1)\beta^{n-2}] \\ &= \tfrac{1}{5}[(\alpha+1)^2 \alpha^{2n-3} + (\beta+1)^2 \beta^{2n-3} - (-1)^n]. \end{aligned}$$

Similarly,

$$a^2 - b^2 = \tfrac{1}{5}[(\alpha+1)^2 \alpha^{2n-3} + (\beta+1)^2 \beta^{2n-3} + 4(-1)^n] = ab + (-1)^n,$$

whence $$a_n^2 = 4ab(a^2 - b^2) + 1 = [2ab + (-1)^n]^2.$$

(III.) Let u_n denote the n-th term of 1, 1, 2, 3, 5, 8, 13, 21, ...; then

$$u_{n+2}^4 - 1 = u_n u_{n+1} u_{n+3} u_{n+4};$$

see Vol. LXXIV., p. 33. Hence

$$\begin{aligned} u_{n+2}^4 + 2u_{n+2}^2 u_{n+1} u_n + u_{n+1}^2 u_n^2 &= 1 + u_n u_{n+1} u_{n+3} u_{n+4} + 2u_{n+2}^2 u_{n+1} u_n + u_{n+1}^2 u_n^2 \\ &= 1 + u_n u_{n+1}[u_{n+3}(u_{n+3} + u_{n+2}) + 2u_{n+2}^2 + (u_{n+3} - u_{n+2})(2u_{n+2} - u_{n+3})] \\ &= 1 + 4u_n u_{n+1} u_{n+2} u_{n+3} = 1 + 4ab(a+b)(a-b) = 4a^3 b - 4ab^3 + 1, \end{aligned}$$

where $a = u_{n+2}$ and $b = u_{n+1}$. Thus

$$a_n = u_{n+2}^2 + u_{n+1} u_n = u_{n+2}(u_{n+1} + u_n) + u_{n+1} u_n = ab + ac + bc,$$

where $c = u_n$. Hence the series a_n is determined; the first five terms are 5, 11, 31, 79, 209.

Digitized by Google

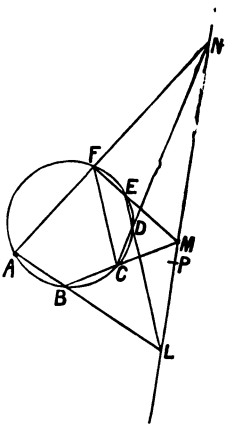
A Corollary to Euclid III. 22.

By the Rev. C. Taylor, D.D.

Let the sides AB, BC, CD of a convex cyclic hexagon meet the opposite sides DE, EF, FA in successive points L, M, N. Then

$$\angle L + \angle N = \angle M.$$

This would be useful in editions of Euclid as a corollary or additional proposition. Assuming it as a lemma, we see that the three circles LDB, MBF, NFD meet in a point, and Pascal's theorem follows at once for a cyclic hexagon.



Case 1. Of the first and third of these circles, let one pass through M. Then the other passes through M; the angles DML, DMN are equal to the supplementary angles DBA, DFA; and LMN is a straight line.

Case 2. Similarly, when the three circles do not meet in M they meet in another point P, and then the angles DPL, DPN are supplementary. Thus P is in line with LN, and so with LM or MN; and therefore LMN is a straight line.

For the three circles we might have used the triad determined in like manner by L, M, N and the other three alternate vertices A, C, E of the hexagon.

The lemma may be proved in different ways:—

(i.) Pascal merely stated his theorem for the circle and then projected. Taking the simplest case, suppose AB, BC parallel to DE, EF. Then the angles B, E are equal; the angles AFC, FCD are equal; and AF, CD are parallel. In the general case, the differences of B, E and AFC, FCD are equal to one another; and they are equal to M−L and N respectively.

Thus I first came upon the lemma, which was new to me.

(ii.) The circular measures of 2L, 2M, and 2N are equal to

$$AF + FE - (BC + CD), \quad AB + AF - (CD + DE) \ \ldots\ldots \ (1, 2),$$
$$AB + BC - (DE + EF) \ \ldots\ldots\ldots\ldots\ (3);$$

and (1) + (3) reduces to (2).

(iii.) Whether the hexagon be cyclic or not, the angle D is equal to the sum of the angles A, L, N, and to that of $\pi - C$, $\pi - E$, M. Therefore $L + N - M$ is equal to $2\pi - (A + C + E)$, which vanishes when the hexagon is cyclic.

This note was suggested by the elementary proof of Pascal's theorem for the case of the circle given by Mr. R. F. Davis in the *Mathematical Reprint*, Vol. LXXIV., p. 112. The use of the lemma as above seems to me to make it somewhat simpler and much easier to remember.

Digitized by Google

15080. (Professor Nanson.)—Lines drawn through the vertices of a triangle Δ divide the angles into the segments α, α'; β, β'; γ, γ', and form a triangle Δ'. If R, R′ are the radii of the circum-circles of Δ, Δ', prove that

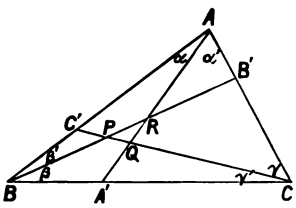
$$\frac{R'}{R} = \frac{\sin\alpha \sin\beta \sin\gamma - \sin\alpha' \sin\beta' \sin\gamma'}{\sin(\beta+\gamma')\sin(\gamma+\alpha')\sin(\alpha+\beta')},$$

and deduce the theorems of Ceva and Menelaus.

Solution by R. Tucker, M.A., *and* Rev. T. Roach, M.A.

$BR = c \sin\alpha/\sin(\alpha+\beta')$,

$BP = a \sin\gamma'/\sin(\beta+\gamma')$;



therefore

$PR \sin(\alpha+\beta')\sin(\beta+\gamma')$

$= c \sin\alpha \sin(\beta+\gamma') - a \sin\gamma' \sin(\alpha+\beta')$

$= 2R\left[\sin(\gamma+\gamma')\sin\alpha\sin(\beta+\gamma')\right.$

$\left. -\sin(\alpha+\alpha')\sin\gamma'\sin(\alpha+\beta')\right]$

$=$ (after reduction)

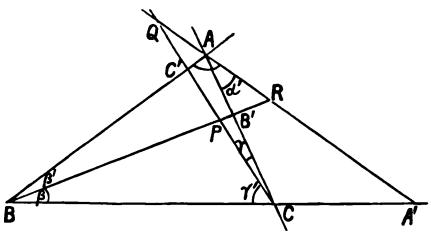
$2R\left[\Pi\sin\alpha - \Pi\sin\alpha' + \sin^2\gamma'\sin\alpha\cos(\beta+\gamma) - \sin^2\alpha\sin\gamma'\cos(\alpha'+\beta')\right.$

$\left. + \sin\alpha\sin\gamma'\cos\gamma'\sin(\beta+\gamma) - \sin\alpha\sin\gamma\cos\alpha\sin(\alpha'+\beta')\right]$

$= 2R\left[\Pi\sin\alpha - \Pi\sin\alpha' + \sin\alpha\sin\gamma'\left\{\sin(\beta+\gamma+\gamma') - \sin(\alpha+\alpha'+\beta')\right\}\right]$

$= 2R\left[\Pi\sin\alpha - \Pi\sin\alpha'\right]$;

therefore $\quad R'/R = \left[\Pi\sin\alpha - \Pi\sin\alpha'\right]/\Pi\sin(\alpha+\beta')$,



R′ vanishes when numerator vanishes, which proves Ceva's and Menelaus's theorems (*cf.* Lachlan, *Modern Pure Geometry*, pp. 52, 58).

15109. (Lt.-Col. Allan Cunningham, R.E.)—Find the least solution in *positive* integers of $x^3+y^3 = mz^3$, where $m = (3\mu-1)$, a positive integer > 2, and $x \neq y$; and x, y, z have no common factor.

Solution by the Proposer.

$$(x^3+y^3) = (x+y)(x^2-xy+y^2) = mz^3.$$

As (x^2-xy+y^2) is known to contain no factors of form $m = 3\mu-1$, the latter must be a factor of $(x+y)$. Hence $x+y = m$, or $= m\lambda$; but the latter will be found (see the Proposer's solution of Question 14945) to

Digitized by Google

involve $x = \lambda\xi$, $y = \lambda\eta$, $z = \lambda\zeta$, an excluded case. Substituting $x+y = m$, and dividing by m, gives

$$m^2 - 3xy = z^3 \quad \text{and} \quad xy = \tfrac{1}{3}(m^2 - z^3) = a \text{ positive } \textit{integer}.$$

Hence also $$(x-y)^2 = \tfrac{1}{3}(4z^3 - m^2) = a\square.$$

These involve $z = 3\zeta + 1$ (since $m = 3\mu - 1$), and $4 > m^2/z^3 > 1$. A few trials now suffice to give the *lowest* solution $m = 20$, $z = 7$, $x-y = 18$, $x+y = 20$, $x = 19$, $y = 1$, and $19^3 + 1^3 = 20 \cdot 7^3$.

5734 & 5702. (5734.) (A. Martin.)—A has m dollars and B n dollars, and they play at a game for which it is $b : a$ that B shall win any one game; prove that the chances for the ultimate success of each player are

$$a^n(b^m - a^m)/(b^{m+n} - a^{m+n}) \text{ and } b^m(b^n - a^n)/(b^{m+n} - a^{m+n}) \text{ respectively.}$$

(5702.) (Rev. Prebendary W. A. Whitworth, M.A.)—A and B play at a game which cannot be drawn, and the odds are $\mu : \nu$ in favour of A winning any assigned game. They agree to play until either A shall have won by scoring m games ahead of B, or B shall have won by scoring n games ahead of A. Show that, when A has scored p games more than B, the odds in favour of his winning the match are

$$\mu^{m+n} - \mu^{m-p}\nu^{n+p} : \mu^{m-p}\nu^{n+p} - \nu^{m+n}.$$

[From this follows as a corollary Ex. 236 in the 3rd Edition of *Choice and Chance*.]

Solutions (5734) *by* H. W. Curjel, M.A., *and* Professor Nanson;
(5702) *by* H. W. Curjel, M.A.

(5734) Let u_m = chance of A's ultimate success when he has m dollars. Then $u_x = a/(a+b)\,u_{x+1} + b/(a+b)\,u_{x-1}$, and $u_0 = 0$ and $u_{m+n} = 1$. Hence

$$u_x = A + B(b/a)^x = (b^x a^{m+n-x} - a^{m+n})/(b^{m+n} - a^{m+n}),$$

from the initial and final conditions; therefore

$$u_m = a^n(b^m - a^m)/(b^{m+n} - a^{m+n}),$$

and B's chance of ultimate success $= 1 - u_m = b^m(b^n - a^n)/(b^{m+n} - a^{m+n})$.

(5702) This follows from the above by putting μ and ν for a and b, and assuming that A and B have $n+p$ and $m-p$ dollars respectively, and we get $u_{n+p} : 1 - u_{n+p} = \mu^{m+n} - \mu^{m-p}\nu^{n+p} : \mu^{m-p}\nu^{n+p} - \nu^{m+n}$.

15104. (R. Knowles.)—Prove that the sum to r terms of the series

$$1 - 10 + 35 - 84 + \ldots \pm \tfrac{1}{2}r(4r^2 - 1)$$

is $-\frac{1}{8}r(4r^2 + 6r - 1)$ when r is even and $\frac{1}{8}(4r^3 + 6r^2 - r - 3)$ when r is odd.

Digitized by Google

Solution by T. Peele, A.I.A.

Using the same notation as in Question 15051,

$$s_p = -\tfrac{1}{3}(48p^2-24p+3), \quad S_{2n} = \sum_{p=1}^{p=n} s_p = -\tfrac{1}{3}n(16n^2+12n-1)$$

Let r be even, $= 2n$ say;

$$S_r = -\tfrac{1}{8}r(4r^2+6r-1);$$

$$S_{2n-1} = S_{2n} + t_{2n} = -\tfrac{1}{3}n(16n^2+12n-1) + \tfrac{2}{3}n(16n^2-1) = \tfrac{1}{3}(16n^3-12n^2-n)$$

Let r be odd, $= 2n-1$ say;

$$S_r = \tfrac{1}{8}(4r^3+6r^2-r-3).$$

14930. (Emile Lemoine.)—Trouver dans un triangle ABC un point M tel que si l'on mène par M une parallèle à BC coupant AB en A_b, AC en A_c; une parallèle à CA coupant BC en B_c, BA en B_a; une parallèle à AB coupant CA en C_a, CB en C_b, les deux triangles $A_bB_cC_a$, $A_cB_aC_b$ soient égaux.

Solution by J. Prescott, B.A.

If M be the centre of gravity of the triangle, M satisfies the required condition. For, let AMX be one of the medians. Then since AM = 2MX, therefore

$$AA_b = 2A_bB$$

and

$$AA_c = 2A_cC,$$

that is, B_a, A, C_b, B_c, A, C_a are the points of trisection of the sides of the triangle. Now $BC_b = \frac{1}{3}BC = \frac{2}{3}BX$; therefore $B_aC_b = \frac{2}{3}AX$. Similarly,

$$C_aB = \tfrac{2}{3}AX;$$

therefore

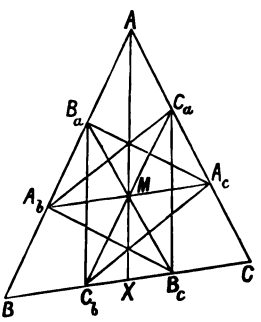
$$B_aC_b = C_aB_c.$$

In the same way

$$A_bC_a = A_cC_b \quad \text{and} \quad B_aA_c = B_cA$$

therefore

$$\triangle A_bB\,C_a = A\,B_aC_b.$$



15118. (Professor Neuberg.)—On mène à deux cercles donnés A, B des tangentes a, b telles que la bissectrice de l'angle ab soit parallèle à une droite donnée m. Trouver le lieu du point de rencontre C des droites a, b, et construire la tangente à ce lieu au point C.

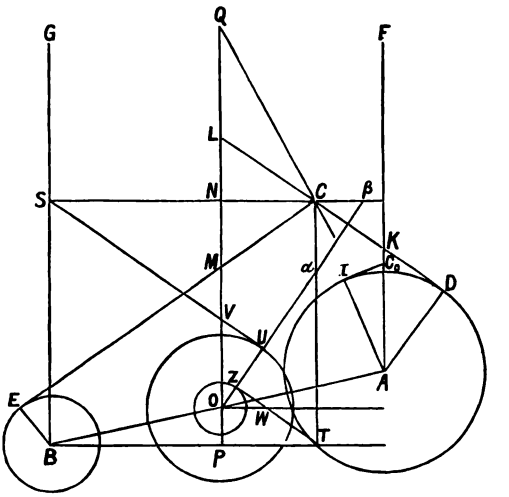
Solutions (I.) *by* D. Biddle; (II.) *by the* Proposer.

(I.) This question comprises three cases, namely, (1) that in which the two circles lie between the two tangents, (2) that in which the two tangents

Digitized by Google

lie between the two circles, (3) that in which one circle is internal and the other external to the tangents. But the principle is much the same in each, and we will therefore consider the first alone.

Let O be the mid-point between the two centres of the given circles.



Draw OQ, AF, BG parallel to the given straight line, and take D, E on the circumferences, such that DAF = EBG = θ, and let the tangents DC, EC meet OQ in L, M respectively. Then

$$\tfrac{1}{2}(OL+OM) = ON = x \quad \text{and} \quad \tfrac{1}{2}(OL-OM)\cot\theta = NC = y.$$

Let BP = p, OP = q; then

$$OL = R\sec\theta + p\tan\theta + q \quad \text{and} \quad OM = r\sec\theta + p\tan\theta - q,$$

where R, r are the radii to A, B respectively. Consequently

$$x = \tfrac{1}{2}(R+r)\sec\theta + p\tan\theta \quad \text{and} \quad y = \tfrac{1}{2}(R-r)\operatorname{cosec}\theta + q\cot\theta.$$

About O describe two circles, of radius $\frac{1}{2}(R+r)$, and $\frac{1}{2}(R-r)$ respectively. Then a simple method of finding a point on the required locus is as follows:—Take U on the larger of the two concentric circles and join OU, cutting the smaller circle in Z. Draw the tangents US, ZT meeting BG, BP in S, T respectively. Then C is found on completing the parallelogram. It is evident that the locus intersects AF at C_0, where the tangent to circle B also touches A, the proper tangent to A being on the opposite side of AF to IC_0.

In order to draw, at C, a tangent to the locus, produce OU to cut TC, SC in α, β respectively, and on OQ find Q, such that

$$QN : NC = S\beta : T\alpha.$$

That QC is tangential to the locus appears from the fact that

$$T\alpha = \left\{\tfrac{1}{2}(R-r)\cot\theta + q\operatorname{cosec}\theta\right\}\operatorname{cosec}\theta$$

and

$$S\beta = \left\{\tfrac{1}{2}(R+r)\tan\theta + p\sec\theta\right\}\sec\theta,$$

Digitized by Google

which, the former taken negatively, are the respective differential coefficients of y, x with respect to θ.

(II.) (i.) J'adopte les axes coordonnés OQ, OW et les notations de M. Biddle. Les coordonnées des centres A, B étant (p, q), $(-p, -q)$, les équations des tangentes DC, EC sont

$$(x-q)\cos\theta + (y-p)\sin\theta = R, \quad (x+q)\cos\theta - (y+p)\sin\theta = r.$$

Tirons-en les valeurs de $\cos\theta$, $\sin\theta$ pour les substituer dans l'égalité $\cos^2\theta + \sin^2\theta = 1$; nous aurons pour le lieu de C une équation du quatrième degré.

(ii.) Décomposons la vitesse V du point C sur sa trajectoire (1°) en deux vitesses v et u dirigées suivant CD et une perpendiculaire à DC; (2°) en deux vitesses v' et u' dirigées suivant CE et une perpendiculaire à EC. Alors les composantes u et u' sont proportionnelles aux longueurs des tangentes DC, EC; car ces tangentes ont des vitesses angulaires égales et de sens contraires. Il résulte de là que

$$\frac{\cos(V, u)}{\cos(V, u')} = \frac{\sin(V, CD)}{\sin(V, CE)} = \frac{CD}{CE};$$

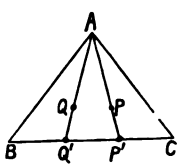
par conséquent la tangente est dirigée suivant la symédiane du triangle CDE.

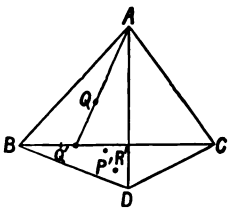
Note on two connected Problems in Mean Value.

By R. Chartres.

(1) Two random points are taken in a triangle and joined to a vertex: find the mean value of the triangle thus formed.

(2) Three random points are taken in a tetrahedron and joined to a vertex: find the mean value of the tetrahedon thus formed.





Let the area of ABC, and the volume of ABCD, be each unity.

(1) The mean value of $AQ'P' = \frac{1}{3}$. Now let Q move in the element AQ', and this result will be reduced to $\frac{2}{3}$ of its value. Therefore mean value of $APQ = (\frac{2}{3})^2$ of $\frac{1}{3} = \frac{4}{27}$.

(2) Since the mean value of $P'Q'R' = \frac{1}{12}BCD$, therefore mean value of $AP'Q'R' = \frac{1}{12}$. Now, let Q move in the element AQ', and this result will be reduced to $\frac{3}{4}$ of its value. Therefore mean value of $APQR = (\frac{3}{4})^3$ of $\frac{1}{12} = \frac{9}{256}$.

Digitized by Google

15150. (Professor LANGHORNE ORCHARD, M.A., B.Sc.)—Find the product of the two series

(1) $1+3+7+13+21+31+\ldots$ to n terms;

(2) $1+13+73+241+601+1261+\ldots$ to n terms.

Solutions (I.) *by* R. W. D. CHRISTIE *and* Rev. T. ROACH, M.A.; (II.) *by* J. BLAIKIE, M.A.

(I.) The general terms are n^2-n+1, n^4-n^2+1.

The sums are $\frac{1}{3}n(n^2+2)$ and $\frac{1}{10}n(2n^4+5n^3-5n+8)$.

The product required is $\frac{1}{30}n^2(n^2+2)(2n^4+5n^3-5n+8)$.

(II.) Series (1)

$= \Sigma\{1+(n-1)n\}$

$= n+\frac{1}{3}(n-1)n(n+1) = \frac{1}{3}n(n^2+2)$.

Series (2)

$= \Sigma\{1+(n^2-1)n^2\}$

$= \Sigma\{1+\frac{1}{3}(n-2)(n-1)n(n+1)+\frac{1}{3}(n-1)n(n+1)(n+2)\}$

$= n+\frac{1}{10}(n-2)(n-1)n(n+1)(n+2)+\frac{1}{10}(n-1)n(n+1)(n+2)(n+3)$

$= \frac{1}{10}n(2n^4+5n^3-5n+8)$;

therefore product of series $= \frac{1}{30}n^2(n^2+2)(2n^4+5n^3-5n+8)$.

3619. (OMEGA.)—Given the sides of three squares inscribed in a triangle, determine the triangle. [Mr. BIDDLE points out that the equations given in the allied Question 9526 are incorrect, but could be readily put into the proper form. See Old Questions in *Ed. Times*, May 1, 1902.]

Solution by D. BIDDLE.

Let $l>m>n$, and let a, b, c be the sides of the required triangle ABC, on which the squares of side l, m, n are respectively based. Then

$$a = l(1+\cot B+\cot C),\quad b = m(1+\cot A+\cot C),$$
$$c = n(1+\cot A+\cot B),$$

and
$$l(1+\cot B+\cot C)/\sin A = m(1+\cot A+\cot C)/\sin B$$
$$= n(1+\cot A+\cot B)/\sin C.$$

Multiply by $\sin A \sin B \sin C$, and reduce; then

$$l(\sin A+\sin B\sin C) = m(\sin B+\sin A\sin C) = n(\sin C+\sin A\sin B),$$

whence
$$\sin A = \frac{\sin B(m-l\sin C)}{l-m\sin C},\quad \sin B = \frac{\sin C(n-m\sin A)}{m-n\sin A},$$
$$\sin C = \frac{\sin A(l-n\sin B)}{n-l\sin B}.$$

From these values we obtain

$$\frac{\sin B}{\sin A} = \frac{b}{a} = \frac{l-m\sin C}{m-l\sin C} \quad\text{and}\quad \frac{\sin C}{\sin A} = \frac{c}{a} = \frac{l-n\sin B}{n-l\sin B},$$

Digitized by Google

in which the lengths of b and c in relation to a are shown to be dependent on the angles formed by b and a, c and a respectively. If a be taken of a *given* length, we can trace, by aid of the two formulæ, two loci of A′, and where these intersect will be the apex of a triangle having the given length as base, and similar in all respects to that required. It will be easy to alter the size to suit the given inscribed squares.

15136. (Professor Cochez.)—Étudier les courbes

(1) $(x-y)^2x+2(x-y)y+1=0$, (2) $y^2=(x^4-1)/x^2$.

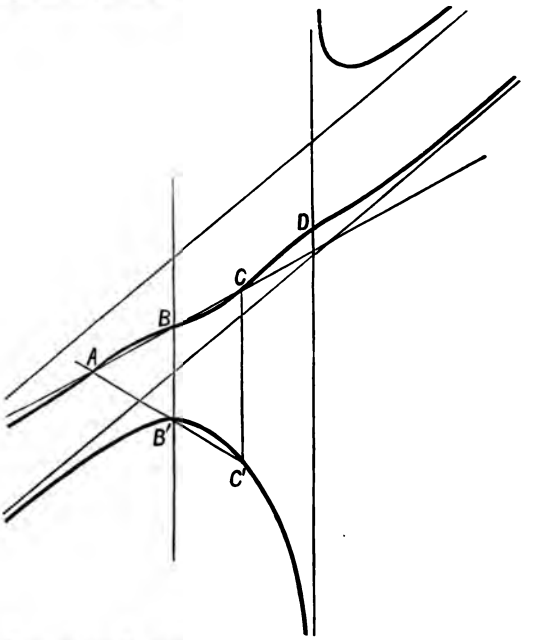
Solution by A. M. Nesbitt, M.A.

(1) The asymptotes are $x-y+2=0$, $x-y=0$, $x+2=0$.

The points B, B′ $(0, \pm 1/\sqrt{2})$, C, C′ $(1, \pm\sqrt{2})$, A $(-1, 0)$, and D $(2, \frac{2}{3})$ are on the curve; and the values of dy/dx (exact or approximate) at these points are B (·68), B′ (−·32), A $(\frac{1}{4})$, C (·77), C′ (−2·77), D (·9).

The three points A, B, C are collinear, as are also A′B′C′.

The serpentine branch is difficult to draw exactly, as the inclinations

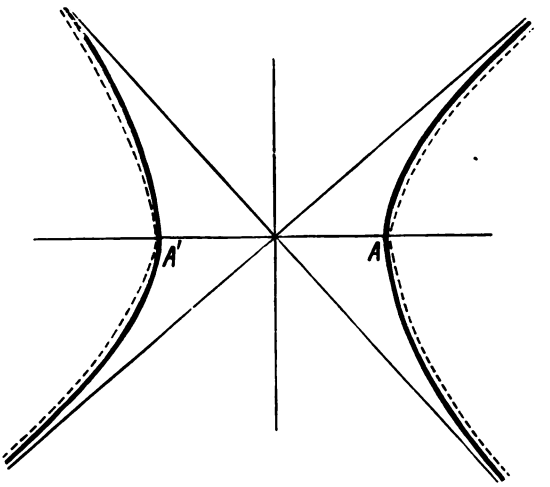


at A, B, C are so nearly equal; but it crosses ABC from the lower side to the upper at A and C.

Digitized by Google

(2) $$x^2y^2 = x^4 - 1.$$

Here $y \pm x = 0$ are asymptotes. Also $x^2 = 0$ theoretically gives the asymptotes. In order to discover precisely what happens at infinity on the axis of y—for the curve has no real points between $x = +1$ and $x = -1$—we may make it homogeneous by introducing z and then put $y = 1$. By this means the old axis of x is sent to infinity, and the old



line at infinity becomes the new axis of z. We get $x^2 = x^4 - z^4$, which yields an imaginary tacnode at the new origin. Hence the original curve has an imaginary tacnode)|(at infinity on the axis of y.

The rest of the curve presents no special difficulty. It lies entirely outside the equilateral hyperbola $x^2 - y^2 = 1$ (dotted in the figure), and has points of undulation at A and A'.

The imaginary tacnode at infinity on the axis of y cannot well be represented in the figure.

15033. (R. Tucker, M.A.)—Three circles are given by

$$x^2 + y^2 + C = 0, \quad x^2 + y^2 + Ax = 0, \quad x^2 + y^2 + By = 0.$$

Find the area of the triangle formed by the polars of their radical centre with respect to the three circles. [Has the question been considered in the general, or any particular form, geometrically or analytically?]

Solution by Professor Sanjána, M.A., *and* Constance I. Marks, B.A.

The radical centre is evidently $x' = C/A$, $y' = C/B$. Its polars with respect to the given circles are

$$xx' + yy' + C = 0, \quad xx' + yy' + \tfrac{1}{2}A(x + x') = 0, \quad xx' + yy' + \tfrac{1}{2}B(y + y') = 0.$$

Digitized by Google

Twice the area of the triangle formed by these

$$= \frac{\left\{-\frac{1}{4}AB\,(x'^2+y'^2)+\frac{1}{2}C\,(Bx'+Ay')+\frac{1}{4}ABC\right\}^2}{\frac{1}{4}ABx'y'\left\{\frac{1}{2}\,(Bx'+Ay')+\frac{1}{4}AB\right\}};$$

see SALMON, § 39. Substitute the values of x' and y', and simplify; we shall get the area $= \dfrac{\left\{C\,(A^2+B^2)+A^2B^2\right\}^2}{2AB\left\{2C\,(A^2+B^2)+A^2B^2\right\}}$.

Professor SANJÁNA remarks :—"I have not seen the Question treated analytically anywhere; in the form given, it does not seem capable of geometrical solution."

5473. (H. W. HARRIS.)—Prove that, if

$$F(m, n, p) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha^m & \beta^m & \gamma^m & \delta^m \\ \alpha^n & \beta^n & \gamma^n & \delta^n \\ \alpha^p & \beta^p & \gamma^p & \delta^p \end{vmatrix},$$

then $F(m, n, p)\left\{F(1, 2, 3)\right\}^2 = \begin{vmatrix} F(m, 2, 3), & F(1, m, 3), & F(1, 2, m) \\ F(n, 2, 3), & F(1, n, 3), & F(1, 2, n) \\ F(p, 2, 3), & F(1, p, 3), & F(1, 2, p) \end{vmatrix}$,

and generalize for a similar determinant of any order.

Solution by Professor NANSON.

If (abc), (xyz) are any two third order determinants, it is known that

$$\begin{vmatrix} (ayz) & (xaz) & (xya) \\ (byz) & (xbz) & (xyb) \\ (cyz) & (xcz) & (xyc) \end{vmatrix} = (abc)(xyz)^2.$$

Further, if H_n is the complete symmetric function of $\alpha\beta\gamma\delta$ of order n, it is known that

$$\frac{F(m, n, p)}{F(1, 2, 3)} = \begin{vmatrix} H_{m-1} & H_{n-1} & H_{p-1} \\ H_{m-2} & H_{n-2} & H_{p-2} \\ H_{m-3} & H_{n-3} & H_{p-3} \end{vmatrix}.$$

It therefore follows that

$$F(mnp)\left\{F(\mu\nu\varpi)\right\}^2 = \begin{vmatrix} F(m\nu\varpi) & F(\mu n\varpi) & F(\mu\nu m) \\ F(n\nu\varpi) & F(\mu n\varpi) & F(\mu\nu n) \\ F(p\nu\varpi) & F(\mu p\varpi) & F(\mu\nu p) \end{vmatrix}.$$

Taking $\mu, \nu, \varpi = 1, 2, 3$, the required result follows. The same proof applies to determinants of any order.

9822. (Professor G. B. M. ZERR.)—If $y = x^{n-1}\log x$, prove that, when r is not $> n$,

$$\frac{d^r y}{dx^r} = (n-1)(n-2)\ldots(n-r)\,x^{n-r-1}\left\{\log x + \left(\frac{1}{n-1}+\frac{1}{n-2}+\ldots+\frac{1}{n-r}\right)\right\};$$

Digitized by Google

and hence show that

$$(1+x)^{n-1}\log(1+x) = x + \frac{(n-1)(n-2)}{1.2}\left\{\frac{1}{n-1}+\frac{1}{n-2}\right\}x^2$$
$$+\frac{(n-1)(n-2)(n-3)}{1.2.3}\left\{\frac{1}{n-1}+\frac{1}{n-2}+\frac{1}{n-3}\right\}x^3+\dots$$
$$+\left\{\frac{1}{n-1}+\frac{1}{n-2}+\frac{1}{n-3}+\dots+\frac{1}{1}\right\}x^{n-1}$$
$$+\frac{x^n}{n}\left\{1-\frac{1.x}{n+1}+\frac{1.2.x^2}{(n+1)(n+2)}-\frac{1.2.3.x^3}{(n+1)(n+2)(n+3}+\dots\right\}.$$

Solution by the PROPOSER.

$$\frac{dy}{dx} = (n-1)x^{n-2}\log x + x^{n-2} = (n-1)x^{n-2}\left\{\log x + \frac{1}{n-1}\right\},$$

$$\frac{d^2y}{dx^2} = (n-1)(n-2)x^{n-3}\log x + (n-1)x^{n-3} + (n-2)x^{n-3}$$
$$= (n-1)(n-2)x^{n-3}\left\{\log x + \left(\frac{1}{n-1}+\frac{1}{n-2}\right)\right\},$$

$$\frac{d^3y}{dx^3} = (n-1)(n-2)(n-3)x^{n-4}\log x + (n-1)(n-2)x^{n-4}$$
$$+(n-1)(n-3)x^{n-4} + (n-2)(n- \qquad ^{n-4}$$
$$= (n-1)(n-2)(n-3)x^{n-4}\left\{\log x + \left(\frac{1}{n-1}+\frac{1}{n-2}+\frac{1}{n-3}\right)\right\}.$$

And, generally,

$$\frac{d^ry}{dx^r} = (n-1)(n-2)\dots(n-r)x^{n-r-1}\left\{\log x + \left(\frac{1}{n-1}+\frac{1}{n-2}+\dots+\frac{1}{n-r}\right)\right\}.$$

Substitute $(x+1)$ for x, and we get

$$\frac{d^ry}{dx^r} = (n-1)(n-2)\dots(n-r)(1+x)^{n-r-1}$$
$$\times\left\{\log(1+x)+\left(\frac{1}{n-1}+\frac{1}{n-2}+\dots+\frac{1}{n-r}\right)\right\}.$$

When $r=n$, we get

$$\frac{d^ny}{dx^n} = (n-1)(n-2)\dots(n-n)(1+x)^{n-n-1}$$
$$\times\left\{\log(1+x)+\left(\frac{1}{n-1}+\frac{1}{n-2}+\dots+\frac{1}{n-n}\right)\right\}$$
$$= (n-1)(n-2)\dots 2.1(1+x)^{-1}.$$

Since all the other values vanish for values of $r =$ or $> n$, we can omit them and proceed on the value of d^ny/dx^n; therefore

$$\frac{d^{n+1}y}{dx^{n+1}} = -1(n-1)(n-2)\dots 2.1(1+x)^{-2},$$

$$\frac{d^{n+2}y}{dx^{n+2}} = 1.2(n-1)(n-2)\dots 2.1(1+x)^{-3},$$

generally $\dfrac{d^{n+m}y}{dx^{n+m}} = (-1)^m.1.2\dots m(n-1)(n-2)\dots 2.1(1+x)^{-(m+1)}$.

Digitized by Google

Now, by MACLAURIN'S formula,

$$y = (1+x)^{n-1}\log(1+x) = f(0) + f_1(0)\frac{x}{1} + f_2(0)\frac{x^2}{1.2} + f_3(0)\frac{x^3}{3!} + \dots$$
$$\dots + f_{n-1}(0)\frac{x^{n-1}}{(n-1)!} + \dots + f_{n+m}(0)\frac{x^{n+m}}{(n+m)!},$$

$$f(0) = 0, \quad f_1(0) = 1, \quad f_2(0) = (n-1)(n-2)\left\{\frac{1}{n-1} + \frac{1}{n-2}\right\},$$

$$f_3(0) = (n-1)(n-2)(n-3)\left\{\frac{1}{n-1} + \frac{1}{n-2} + \frac{1}{n-3}\right\},$$

$$f_{n-1}(0) = (n-1)(n-2)\dots 2.1\left\{\frac{1}{n-1} + \frac{1}{n-2} + \dots \frac{1}{1}\right\},$$

$$f_{n+m}(0) = (-1)^m\, 1.2\dots m\,(n-1)(n-2)\dots 2.1.$$

Hence, substituting in MACLAURIN'S formula,

$$(1+x)^{n-1}\log(1+x) = x + \frac{(n-1)(n-2)}{1.2}\left\{\frac{1}{n-1} + \frac{1}{n-2}\right\}x^2$$
$$+ \frac{(n-1)(n-2)(n-3)}{1.2.3}\left\{\frac{1}{n-1} + \frac{1}{n-2} + \frac{1}{n-3}\right\}x^3 + \dots$$
$$\dots + \left\{\frac{1}{n-1} + \frac{1}{n-2} + \dots + \frac{1}{1}\right\}x^{n-1} + \frac{x^n}{n} - \frac{x^{n+1}}{n(n+1)}$$
$$+ \frac{1.2.x^{n+2}}{n(n+1)(n+2)} - \frac{1.2.3.x^{n+3}}{n(n+1)(n+2)(n+3)} + \dots,$$

which is the stated result.

14883. (The late Professor WOLSTENHOLME, M.A., Sc.D.)—Prove that $\int_0^\infty \frac{\log x}{1-x^n}dx = -\left(\frac{\pi}{n\sin\pi/n}\right)^2$, $\int_0^\infty \frac{\log x}{1+x^n}dx = -\cos\frac{\pi}{n}\left(\frac{\pi}{n\sin\pi/n}\right)^2$

$$\int_0^\infty \frac{(\log x)^3}{1-x^n}dx = -2\left(\frac{\pi}{n\sin\pi/n}\right)^4\left\{1 + 2\cos^2\pi/n\right\},$$

$$\int_0^\infty \frac{(\log x)^3}{1+x^n}dx = -\left(\frac{\pi}{n\sin\pi/n}\right)^4(5\cos\pi/n + \cos^3\pi/n),$$

n being greater than 1; and generally that

$$\int_0^\infty \frac{(\log x)^{2r+1}}{1-x^n}dx = -\theta^{2r+2}\left(\frac{d}{d\theta}\right)^{2r}\left(\frac{1}{\sin^2\theta}\right),$$

and

$$\int_0^\infty \frac{(\log x)^{2r+1}}{1+x^n}dx = -\theta^{2r+2}\left(\frac{d}{d\theta}\right)^{2r+1}\left(\frac{1}{\sin\theta}\right),$$

where $\theta = \pi/n$.

Solution by the PROPOSER.

$$\int_0^\infty \frac{(\log x)^p\,dx}{1-x^n} = \int_0^1 \dots + \int_1^\infty \dots,$$

and, if in the second term we put $1/x$ for x, we get, when p is odd,

$$\int_0^1 \frac{1+x^{n-2}}{1-x^n}(\log x)^p\,dx$$

Digitized by Google

or $\int_0^1 (1 + x^{n-2} + x^n + x^{2n-2} + x^{2n} + x^{3n-2} + \dots \text{ ad inf.})$

$$= \int_0^1 (1 + x^{n-2} + x^n + x^{2n-2} + x^{2n} + x^{3n-2} + \dots \text{ ad inf.})(\log x)^p \, dx$$

$$= (-1)^p p! \left\{ 1 + \frac{1}{(n-1)^{p+1}} + \frac{1}{(n+1)^{p+1}} + \frac{1}{(2n-1)^{p+1}} + \frac{1}{(2n+1)^{p+1}} + \dots \text{ ad inf.} \right\},$$

or, if $p = 2r+1$,

$$= -(2r+1)! \left\{ 1 + \frac{1}{(n-1)^{2r+2}} + \frac{1}{(n+1)^{2r+2}} + \dots \right\}.$$

But
$$\frac{1}{\sin^2\theta} = \frac{1}{\theta^2} + \frac{1}{(\theta-\pi)^2} + \frac{1}{(\theta+\pi)^2} + \dots;$$

$$\therefore \left(\frac{d}{d\theta}\right)^{2r} \left(\frac{1}{\sin^2\theta}\right) = (2r+1)! \left\{ \frac{1}{\theta^{2r+2}} + \frac{1}{(\theta-\pi)^{2r+2}} + \frac{1}{(\theta+\pi)^{2r+2}} + \dots \text{ ad inf.} \right\},$$

$$\theta^{2r+2} \left(\frac{d}{d\theta}\right)^{2r} \left(\frac{1}{\sin^2\theta}\right) = (2r+1)! \left\{ 1 + \left(\frac{\theta}{\theta-\pi}\right)^{2r+2} + \left(\frac{\theta}{\theta+\pi}\right)^{2r+2} + \dots \right\}$$

$$= (2r+1)! \left\{ 1 + \left(\frac{1}{n-1}\right)^{2r+2} + \left(\frac{1}{n+1}\right)^{2r+2} + \dots \right\},$$

whence $\int_0^\infty \frac{(\log x)^{2r+1}}{1-x^n} dx = -\theta^{2r+2} \left(\frac{d}{d\theta}\right)^{2r} \left(\frac{1}{\sin^2\theta}\right), \quad \theta = \frac{\pi}{n}.$

Similarly, $\int_0^\infty \frac{(\log x)^{2r+1}}{1+x^n} dx = \int_0^1 \frac{1-x^{n-2}}{1+x^n} (\log x)^{2r+1} dx,$

and, expanding as before,

$$= -(2r+1)! \left\{ 1 - \frac{1}{(n-1)^{2r+2}} - \frac{1}{(n+1)^{2r+2}} + \frac{1}{(2n-1)^{2r+2}} + \frac{1}{(2n+1)^{2r+2}} - \dots \right\}.$$

But
$$\frac{1}{\sin\theta} = \frac{1}{\theta} + \frac{1}{\pi-\theta} - \frac{1}{\pi+\theta} - \frac{1}{2\pi-\theta} + \frac{1}{2\pi+\theta} + \frac{1}{3\pi-\theta} - \dots,$$

whence $\left(\frac{d}{d\theta}\right)^{2r+1} \left(\frac{1}{\sin\theta}\right) = -(2r+1)! \left\{ \frac{1}{\theta^{2r+2}} - \frac{1}{(\pi-\theta)^{2r+2}} - \frac{1}{(\pi+\theta)^{2r+2}} + \frac{1}{(2\pi-\theta)^{2r+2}} + \frac{1}{(2\pi+\theta)^{2r+2}} - \dots \right\},$

or $\theta^{2r+2} \left(\frac{d}{d\theta}\right)^{2r+1} \left(\frac{1}{\sin\theta}\right)_{\theta=\pi/n}$

$$= -(2r+1)! \left\{ 1 - \frac{1}{(n-1)^{2r+2}} - \frac{1}{(n+1)^{2r+2}} + \frac{1}{(2n-1)^{2r+2}} + \frac{1}{(2n+1)^{2r+2}} - \dots \right\}$$

$$= \int_0^\infty \frac{(\log x)^{2r+1}}{1+x^n} dx.$$

Hence
$$\int_0^\infty \frac{\log x}{1-x^n} dx = -\left(\frac{\pi}{n \sin \pi/n}\right)^2,$$

$$\int_0^\infty \frac{\log x}{1+x^n} dx = -\cos\frac{\pi}{n} \left(\frac{\pi}{n \sin \pi/n}\right)^2,$$

$$\int_0^\infty \frac{(\log x)^3}{1-x^n} dx = -\left(\frac{\pi}{n}\right)^4 \frac{d}{d\theta}\left(-\frac{2\cos\theta}{\sin^3\theta}\right) = -\left(\frac{\pi}{n}\right)^4 \left\{ \frac{2\sin^2\pi/n + 6\cos^2\pi/n}{(\sin \pi/n)^4} \right\}$$

$$= -2 \left(\frac{\pi}{n \sin \pi/n}\right)^4 \left(1 + 2\cos^2\frac{\pi}{n}\right),$$

Digitized by Google

$$\text{and } \int_0^\infty \frac{(\log x)^3}{1+x^n}\,dx = \left(\frac{\pi}{n}\right)^4 \frac{d}{d\theta}\left(\frac{\sin^2\theta + 2\cos^2\theta}{\sin^3\theta}\right)$$

$$= \left(\frac{\pi}{n}\right)^4 \left\{\frac{-2\cos\pi/n\sin^2\pi/n - 3\cos\pi/n\,(1+\cos^2\pi/n)}{(\sin\pi/n)^4}\right\}$$

$$= -\left(\frac{\pi}{n\sin\pi/n}\right)^4 (5\cos\pi/n + \cos^3\pi/n).$$

It has been assumed that n is greater than 1, in finding

$$\int_0^1 x^{n-2}(\log x)^p\,dx,$$

which will be infinite if n be equal to or less than 1.

From the result

$$\int_0^\infty \frac{\log x}{1+x^n}\,dx = -\cos\frac{\pi}{n}\left(\frac{\pi}{n\sin\pi/n}\right)^2 = \int_0^\infty \frac{x^{n-2}}{1+x^n}\log x\,dx,$$

we get, integrating with respect to n,

$$\int_0^\infty \log\left(\frac{1+x^n}{1+x^m}\right)\frac{dx}{x^2} = \pi\left(\frac{1}{\sin\pi/m} - \frac{1}{\sin\pi/n}\right),$$

where both m and n must be greater than 1, and, similarly,

$$\int_0^\infty \log\left(\frac{1-x^n}{1-x^m}\right)\frac{dx}{x^2} = \pi(\cot\pi/m - \cot\pi/n).$$

So also $\int_0^\infty (\log x)^2 \log\left(\frac{1+x^n}{1+x^2}\right)\frac{dx}{x^2}$

$$= \frac{\pi}{\sin\pi/n}\left\{2 + 2\pi/n\cot\pi/n + \pi^2/n^2(1+2\cot^2\pi/n)\right\} - \pi\left(2+\tfrac{1}{6}\pi^2\right),$$

$$\int_0^\infty (\log x)^2 \log\left(\frac{1-x^n}{1-x^2}\right)\frac{dx}{x^2}$$

$$= \pi^2 - \frac{2\pi}{\sin\pi/n}\left\{\cos\frac{\pi}{n} + \frac{\pi}{n\sin\pi/n} + \cos\frac{\pi}{n}\left(\frac{\pi}{n\sin\pi/n}\right)^2\right\}.$$

14917, 15001, 15013, 15073. (A. M. Nesbitt, M.A.)—(14917.) Three conics S_1, S_2, S_3 have a common point O. The remaining intersections of S_2S_3 are the vertices of a triangle Δ_1, those of S_3S_1 of a triangle Δ_2, and those of S_1S_2 of a triangle Δ_3. Prove that the nine sides of these three triangles all touch the same conic U. Or, reciprocally: Given three parabolas, their common tangents, taking the curves in pairs, will form three triangles (one for each pair) whose nine vertices lie on a conic.

(15001.) Two conics U and V intersect in ABCO, and through O two arbitrary straight lines OP, OQ are drawn; OP cuts U in P and V in Q′; OQ cuts U in P′ and V in Q. Prove that the four straight lines OP, OQ, PP′, QQ′ and the three sides of the triangle ABC all touch one and the same conic.

(15013.) If there be three parabolas P, Q, R, and if the common tangents of Q and R form a triangle A, those of R and P a triangle B, and those of P and Q a triangle C, prove that the nine vertices of A, B, C are

Digitized by Google

on a conic (which may be called the "nine-point conic" of the three parabolas). Show also that the four "nine-point conics" of four parabolas (taking them three at a time) pass through a common point. This theorem is of course also true, if for parabolas we substitute "conics having a common tangent," but the enunciation becomes of necessity more verbose.

(15073.) (See Question 14917.)—If we call the conic which touches the nine sides of the triangles Δ_1, Δ_2, Δ_3 the "nine-tangent conic" of the three original conics, prove that, if *four* conics P, Q, R, S be such that all pass through one point, while not more than two pass through any other point, then the four "nine-tangent conics" of QRS, RSP, SPQ, and PQR have a common tangent. The following particular case of the reciprocal of Question 14917 ought to admit of a solution by purely geometrical methods. Two parabolas P, P′ have a common focus S, and a circle C touches their common tangent. If the common tangents to P and C and to P′ and C form triangles Δ and Δ', prove that the point S and the six vertices of Δ and Δ' all lie on a circle.

Solution by Professor Nanson.

The envelope of a line cut in involution by three conics S_1, S_2, S_3 is a class cubic which when the conics have a common point O degenerates into the point O and a conic C_4, which therefore touches the nine sides of Δ_1, Δ_2, Δ_3. If then we have four conics S_1, S_2, S_3, S_4 through O, there are four nine-line conics C_1, C_2, C_3, C_4. Any line which touches two of these conics C_1, C_2 (and there are four such lines), being cut in involution by S_2, S_3, S_4 and by S_1, S_3, S_4, is also (unless the line be a common chord of S_3, S_4) cut in involution by all four S_1, S_2, S_3, S_4, and therefore touches C_3, C_4. Now, exclusive of chords through O, the conics S_3, S_4 have three common chords. Hence C_1, C_2, C_3, C_4 have one common tangent.

This gives 15073 and first part of 14917. Reciprocation then gives 15013 and second part of 14917. Taking one conic to be a pair of straight lines through O, 15001 follows from 14917.

14875. (Lt.-Col. Allan Cunningham, R.E.)—Show that

$$N_1 = x^8 + mx^4y^4 + y^8$$

is resolvable into factors when $m = (\frac{1}{4}\mu^4 - 2\mu^2 + 2)$; also that

$$N_2 = x^8 + x^6y^2 + mx^4y^4 + x^2y^6 + y^8$$

is resolvable into factors when $m = \frac{1}{4}(\mu^2 - 3)^2$.

Ex.— $N_1 = 100002501$, $N_2 = 101090101$.

Digitized by Google

Solutions (I.) *by* Professor SANJÁNA, M.A.; (II.) *by the* PROPOSER.

(I.) Assuming

$$x^8 + mx^4y^4 + y^8 \equiv (x^4 + \mu x^3y + hx^2y^2 + \mu xy^3 + y^4)(x^4 - \mu x^3y + kx^2y^2 - \mu xy^3 + y^4),$$

and equating coefficients, we get

$$k + h - \mu^2 = 0, \quad \mu(k - h) = 0, \quad \text{and} \quad 2 - 2\mu^2 + hk = m;$$

whence $$h = k = \tfrac{1}{2}\mu^2 \quad \text{and} \quad m = \tfrac{1}{4}\mu^4 - 2\mu^2 + 2.$$

When $x = 10$, $y = 1$, $\mu^2 = 1$, we get $h = \frac{1}{2}$, $m = \frac{1}{4}$; thus

$$N_1 = (10000 + 1000 + 50 + 10 + 1)(10000 - 1000 + 50 - 10 + 1)$$
$$= 11061 \cdot 9041,$$

and the prime factors are easily found.

Another value of m is possible, for, assuming

$$x^8 + mx^4y^4 + y^8 \equiv (x^4 + \mu x^3y + hx^2y^2 - \mu xy^3 + y^4)(x^4 - \mu x^3y + kx^2y^2 + \mu xy^3 + y^4),$$

we shall get, finally, $h = k = \frac{1}{2}\mu^2$, $m = \frac{1}{4}\mu^4 + 2\mu^2 + 2$. Thus, when $x = 10$, $y = 1$, $\mu^2 = 1$, $-h = \frac{1}{2}$ and $m = 4\frac{1}{4}$; and we have

$$N'_1 \text{ or } 100042501 = 11041 \cdot 9061.$$

Proceeding similarly, we shall find, in the case of

$$x^8 + x^6y^2 + mx^4y^4 + x^2y^6 + y^8, \quad h = k = \tfrac{1}{2}(\mu^2 + 1) \quad \text{and} \quad m = \tfrac{1}{4}(\mu^2 - 3)^2;$$

so that $N_2 = 13531 \cdot 7471$. The other form of resolution will give $h = \frac{1}{2}(\mu^2 + 1)$ and $m = \frac{1}{4}(\mu^2 + 10\mu + 9)$; so that, taking $\mu = 9$, we get $h = 5$ and $m = 45$; and the number

$$N'_2 \text{ or } 101450101 = 12471 \cdot 7531.$$

(II.) Let $$N = (x^4 + kx^2y^2 + y^4)^2 - \mu^2x^2y^2(x^2 + y^2)^2 = P^2 - Q^2 \ldots\ldots\ldots\ldots(1).$$

Then $$N = x^8 + (2k - \mu^2)x^6y^2 + (k^2 - 2\mu^2 + 2)x^4y^4 + (2k - \mu^2)x^2y^6 + y^8 \ldots\ldots(2).$$

Here the form N, being a difference of squares, is always factorisable.

Case i.—N reduces to N_1 by taking $k = \frac{1}{2}\mu^2$, which gives

$$m = (\tfrac{1}{4}\mu^4 - 2\mu^2 + 2),$$

as required.

Case ii.—N reduces to N_2, by taking $k = \frac{1}{2}(\mu^2 + 1)$, which gives

$$m = \tfrac{1}{4}(\mu^2 + 1)^2 - 2\mu^2 + 2 = \tfrac{1}{4}(\mu^2 - 3)^2,$$

as required.

Ex. i.—$N_1 = 10^8 + \frac{1}{4} \cdot 10^4 + 1$. Here $x = 10$, $y = 1$, $m = \frac{1}{4}$; giving $\mu = 1$, $k = \frac{1}{2}$; therefore

$$N = (10^4 + \tfrac{1}{2} \cdot 10^2 + 1)^2 - 10^2(10^2 + 1)^2 = 10051^2 - 1010^2 = 9041 \cdot 11061.$$

Ex. ii.—$N_2 = 10^8 + 10^6 + 9 \cdot 10^4 + 10^2 + 1$. Here $x = 10$, $y = 1$, $m = 9$; giving $\mu = 3$, $k = 5$; therefore

$$N = (10^4 + 5 \cdot 10^2 + 1)^2 - 3^2 \cdot 10^2(10^2 + 1)^2 = 10501^2 - 3030^2 = 7471 \cdot 13531.$$

Note. — The two extra forms $m = (\frac{1}{4}\mu^4 + 2\mu^2 + 2)$ in N_1 and $m = \frac{1}{4}(\mu^2 + 10\mu + 9)$ in N_2, noticed by Professor SANJÁNA, will be produced by writing $(x^2 - y^2)$ for $(x^2 + y^2)$ in equation (1) of the PROPOSER'S Solution (II.).

Digitized by Google

9773. (B. Peirce.)—A un certain jeu, le joueur A. gagne un point chaque fois qu'il tire une boule blanche d'une urne qui contient 3 boules blanches et 2 boules noires; chaque fois qu'il tire une boule noire, il perd tous ces points, et B., qui ne fait jamais de tirage, en gagne un. Quelle est la probabilité pour A. de faire h points le premier? Cas particulier où $h = 3$.

Solution by H. M. Taylor, M.A., F.R.S.

Let a and $1-a$ be the respective chances of A. and B. gaining a point at any specified drawing. Let $f(r)$ be the chance of B. winning the set when he has just gained his r-th point. The chance then that B. gains his $(r+1)$-th point equals the chance that A. does not gain h points consecutively $= 1-a^h$. Hence

$$f(r) = (1-a^h)f(r+1).$$

Now $f(h) = 1$; therefore $f(r) = (1-a^h)^{h-r}$. Similarly, we see that, if $f(p, r)$ be B.'s chance of winning the set at the time when B. has gained r points and A. has gained p points, then

$$f(p, r) = (1-a^{h-p})f(r+1),$$

and A.'s chance then of winning the set is $1-f(p, r)$. In the special case of the Question $a = \frac{3}{5}$ and $h = 3$. Hence when no drawing has yet taken place A.'s chance of winning the set to B.'s chance is 1011933 to 941192.

Problems whose Arithmetical Solutions are indicated by a Knowledge of Algebra.

By W. R. Worthington, M.A.

Lemma.—Let a and b be two numbers, a the greater. Given their difference $a-b$, and the difference between their reciprocals $1/b-1/a$, to find a and b.

$$1/b-1/a = (a-b)/ab, \quad (a-b) \div (a-b)/ab = ab,$$

$$ab+[\tfrac{1}{2}(a-b)]^2 = [\tfrac{1}{2}(a+b)]^2, \quad \tfrac{1}{2}(a+b) \pm \tfrac{1}{2}(a-b) = a \text{ and } b.$$

So that, (i.) if the difference between two numbers is divided by the difference between their reciprocals, the quotient is the product of the numbers; (ii.) if the product be increased by the square of half the difference between the numbers, the total is the square of half the sum of the numbers; (iii.) the sum and difference of the roots of the two squares are the two numbers.

Let $$a-b = 12 \quad \text{and} \quad (a-b)/ab = \tfrac{1}{240},$$

$$12 \div \tfrac{1}{240} = 2880; \quad 2880+6^2 = 2916 = (54)^2,$$

$$54 \pm 6 = 60 \quad \text{and} \quad 48 = a \text{ and } b.$$

Ex.—A bicyclist, being asked at what rate he had ridden a certain distance, replied that, if he had taken 48 seconds longer over a mile the whole way, he would have lost $2\frac{1}{2}$ miles per hour. Required his rate of riding.

Digitized by Google

A body moving at the rate of (i.) a miles per hour moves 1 mile in $1/a$-th of an hour, (ii.) b miles per hour moves 1 mile in $1/b$-th of an hour; and therefore $1/a \sim 1/b$ is the fraction of an hour lost or gained in 1 mile by the substitution of one rate for the other.

In the question the difference of 48 seconds or $\frac{1}{75}$ hour in a mile makes a difference of $2\frac{1}{2}$ miles in an hour.

$$2\tfrac{1}{2} \div \tfrac{1}{75} = \tfrac{2}{1}\tfrac{7}{2}\tfrac{5}{1} = \tfrac{7}{4}\tfrac{5}{1}\tfrac{0}{0} = \tfrac{3}{2}\tfrac{0}{1} \times \tfrac{2}{1}\tfrac{5}{2}.$$

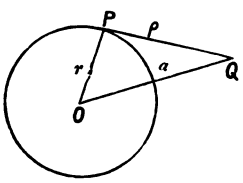
Ans. 15 miles an hour.

15183. (A. F. VAN DER HEYDEN, M.A.)—If a mass m be placed at the centre of a sphere S, the potential due to m, at any point external to S, will be equal to that due to a distribution over S equal in sum to m. Hence deduce that, if a surface S include any number of particles, and if N be the change in the normal force due to these particles on crossing the surface, a distribution over S of density $N/4\pi$ will produce the same potential throughout all space external to S as the masses referred to.

Solution by the PROPOSER.

If a surface S include a number of particles $m_1, m_2, \ldots, m_i, \ldots$, and if N be the change in the normal force due to these masses in crossing the surface, a distribution over the surface S of density $N/4\pi$ will produce the same potential throughout all space external to S as the masses referred to.

(1) Let S be a sphere enclosing a single particle m at its centre O. Let Q be an external point, P a point on the sphere.



$$OP = r : OQ = a : PQ = \rho.$$

Density of distribution $= m/4\pi r^2$.

Potential at Q $= \iint \frac{m\,d\sigma}{4\pi r^2 \rho}$,

where $\iint d\sigma$ denotes integration over the sphere. Taking OQ as axis of spherical coordinates, we get

$$\iint \frac{d\sigma}{\rho} = \int_0^{\pi}\int_0^{2\pi} \frac{r^2 \sin\theta\, d\theta\, d\phi}{(a^2+r^2-2ar\cos\theta)^{\frac{1}{2}}} = 2\pi \int_0^{\pi} \frac{r^2 \sin\theta\, d\theta}{(a^2+r^2-2ar\cos\theta)^{\frac{1}{2}}} = \frac{4\pi r^2}{a};$$

therefore potential at Q $= m/a =$ potential due to mass m.

(2) Let the surface S be any closed surface, including any number of particles.

It follows from the preceding that we may replace any particle by a spherical distribution equal in amount to the mass of the particle. Let this be done for each point, then for each point on each sphere thus obtained, and so on, the radius of each sphere being such that its least distance from S is a small quantity ϵ. We shall thus finally obtain a system of small spheres whose centres are at distances from the given surface S,

Digitized by Google

each less than some quantity ϵ' which may be made as small as we please by sufficiently diminishing ϵ.

Now, let each such sphere be replaced by a particle at its centre. This gives us a surface which will ultimately coincide with the surface S.

Moreover, the potential over the surface and throughout external space will, by the preceding, remain unaltered. Accordingly, the variation N in the normal force, in crossing the surface S, will be the same as before.

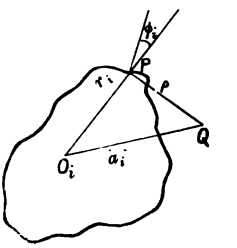
Let d be the density of the new distribution at any point. Then

$$N = 4\pi d;$$

therefore $$d = N/4\pi.$$

The analytical statement of the general question is as follows:

$$\iint \sum_i \frac{m_i \cos\phi_i}{4\pi r_i^2 \rho}\, d\sigma = \sum_i \frac{m_i}{a_i}.$$



15058. (Professor Nanson.)—If OP, OQ are small equal arcs of a curve and its circle of curvature at O, show that the difference of the chords of these arcs is $\frac{1}{24}\kappa\kappa' OP^4$, where κ is the curvature and $\kappa' = d\kappa/ds$.

Solutions (I.) *by* Professor Sanjána, M.A.; (II.) *by* A. M. Nesbitt, M.A.

(I.) Refer the curve to the tangent and normal at O as axes, and let the co-ordinates of P be x, y. Then

$$x = s - s^3/6\rho^2, \quad y = s^2/2\rho - (s^3/6\rho^2)(d\rho/ds) \text{ approximately}$$

(J. Edwards, § 357). Hence

$$(\text{chord OP})^2 = s^2 - \tfrac{1}{3}s^4/\rho^2 + \tfrac{1}{4}s^4/\rho^2 \text{ (approximately)} = s^2 - \tfrac{1}{12}s^4/\rho^2;$$

therefore $$OP = s - \tfrac{1}{24}s^3/\rho^2.$$

Also chord $$OQ = 2\rho \sin\tfrac{1}{2}\theta = \rho\theta - \tfrac{1}{24}\rho\theta^3 = s - \tfrac{1}{24}s^3/\rho^2.$$

Thus chord OQ − chord OP $= -\tfrac{1}{24}s^3/\rho^2$. But

$$\kappa = \rho^{-1}, \quad \kappa' = -\rho^{-2}\, d\rho/ds = -\rho^{-2}\,\rho/s \text{ (approximately)} = -(1/\rho s);$$

hence $$\tfrac{1}{24}\kappa\kappa' OP^4 = \tfrac{1}{24}\frac{1}{\rho}\frac{(-1)}{\rho s}\, s^4 = -\tfrac{1}{24}\, s^3/\rho^2.$$

(II.) Let (x, y) be co-ordinates of P, (ξ, η) co-ordinates of Q. Then x, y are functions of s in the curve, and ξ, η of s in the circle; therefore we may put, accents denoting differentiation for s,

$$x = x_0 + x_0' s + x_0'' \tfrac{1}{2}s^2 + \dots.$$

But $$x_0' = (dx/ds)_0 = (\cos\psi)_0 = 1, \quad x_0'' = [-\sin\psi\,(1/\rho)]_0 = 0,$$
$$x_0''' = [(-\cos\psi + \sin\psi\rho')/\rho^2]_0 = -1/\rho^2,$$
$$x_0'''' = [3\cos\psi\rho'/\rho^3 + \sin\psi\,(\dots)]_0 = 3\rho'/\rho^3.$$

Digitized by Google

Now in the circle the only change is that $\rho' = 0$; therefore

$$x - \xi = \frac{3\rho'}{\rho^3}\tfrac{1}{24}s^4.$$

Again $y = y_0 + y_0's + \ldots$ and $y_0 = 0$; also $\eta_0 = 0$,

$$y_0' = (\sin\psi)_0 = 0 \quad \text{and} \quad \eta_0' = 0,$$

$$y_0'' = [(\cos\psi)/\rho]_0 = 1/\rho \quad \text{and} \quad \eta_0'' = 1/\rho,$$

$$y_0''' = [(-\sin\psi - \cos\psi\rho')/\rho^2]_0 = -\rho'/\rho^2; \quad \text{therefore } \eta_0''' = 0,$$

$$y_0'''' = [(-\cos\psi - \cos\psi\rho\rho'')/\rho^3 + 2\cos\psi\rho'^2/\rho^3 + \text{terms involving } \sin\psi]_0$$
$$-(-1 + 2\rho'^2 - \rho\rho'')/\rho^3; \quad \text{therefore } \eta'''' = -1/\rho^3.$$

Now the difference between the chords OP and OQ

$$= \sqrt{(x^2+y^2)} - \sqrt{(\xi^2+\eta^2)} = x - \xi + \tfrac{1}{2}(y^2/x - \eta^2/\xi),$$

and

$$y = s^2/2\rho - s^3\rho'/6\rho^2 + \ldots \text{ higher powers of } s,$$
$$\eta = s^2/2\rho + 4\text{th and higher powers of } s,$$

while x and ξ both $= s +$ cubes and higher powers; therefore

$$\tfrac{1}{2}(y^2/x - \eta^2/\xi) = -s^4\rho'/12\rho^3.$$

Hence the difference between the chords $= s^4\rho'/24\rho^3$, and, if $\kappa = 1/\rho$, $\kappa' = -\rho'/\rho^2$, therefore the difference $= \frac{1}{24}\mathrm{OP}^4\kappa\kappa'$.

15068. (R. W. D. Christie.)—Solve generally the following Christmas puzzle:—Find four integers a, b, c, d, such that

(1) $ad = bc = 1 \mod p$, (2) $a^2 + b = c + d^2 = 0 \mod p$,

(3) $a^2c + 1 = bd^2 + 1 = 0 \mod p$, (4) $bc^2 + d^2 = 0 \mod p$,

where p = prime of form $4m-1$, and give *all* the solutions.

Solution by Lt.-Col. Allan Cunningham, R.E.

Out of the *seven* congruences to be simultaneously satisfied, only *three* are independent, *e.g.*, $ad \equiv 1$, $bc \equiv 1$, $a^2 + b \equiv 0$ will be found to involve the rest. The number of *incongruous* solutions in + integers, *all* $< p$, will first be found.

One obvious solution is $a = d = 1$, $b = c = p-1$. By (2), b, c must both be 2-ic *non-residues* of p; and have therefore $\frac{1}{2}(p-1)$ values; these must be *paired* together so as to satisfy $bc \equiv 1$. One of these ways is $b = c = p-1$; excluding this, it is known that the remaining $\frac{1}{2}(p-3)$ quantities can be paired together in $\frac{1}{4}(p-3)$ ways, so as to satisfy $bc \equiv 1$, wherein $b \neq c$.

Next, by (2), every value of b gives *two* values of a; and its conjugate c gives two values of d: these values of a, d will be found to form *two* conjugate pairs satisfying $ad \equiv 1$, as required.

Thus each of the $\frac{1}{4}(p-3)$ conjugate pairs of b, c yields *two* conjugate pairs of a, d, wherein $a \neq d$; this gives therefore $\frac{1}{2}(p-3)$ solutions, to

Digitized by Google

which must be added the special solution $a = d = 1$, $b - c = p-1$; total $\frac{1}{2}(p-1)$ solutions in + integers, *all* $< p$, and *incongruous*.

An example will make this clear; take $p = 19$.

$b =$	2 ;	3 ;	8 ;	14 ;	18 ;
$c =$	10 ;	13 ;	12 ;	15 ;	18 ;
$a =$	6, 13 ;	4, 15 ;	7, 12 ;	9, 10 ;	1 ;
$d =$	16, 3 ;	5, 14 ;	11, 8 ;	17, 2 ;	1 ;

} total 9.

The *total* number of solutions (asked for in Question) is, however, *infinite*; for, if (a', b', c', d') be a *particular* solution, so are $(\pm lp + a', \pm mp + b', \pm np + c', \pm rp + d')$.

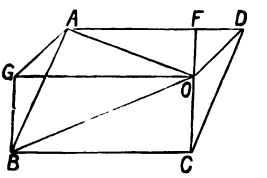
9946. (Professor Genese, M.A.)—Through a point within a parallelogram four circles are described containing the sides as chords. If any two of these circles through the same vertex be equal, all four are.

Solutions (I.) *by* A. M. Nesbitt, M.A., *and* J. H. Taylor, M.A.; (II.) *by* C. A. Laisant.

(I.) If two adjacent circles, say OAB, OBC be equal (but not coincident), we must have $\angle OAB = OCB$. Therefore also $\angle OAD = OCD$, and the other two circles are also equal. Draw OG equal and parallel to BC. Therefore

$$\angle OGB = OCB = OAB,$$

or the four points O, A, G, B are concyclic.



Hence $\angle COD$ = AGB = supplement of AOB, or circle AOB = circle COD.

N.B.—If we are given circle AOB = circle COD, and $\angle AOB + COD$ = two right angles, a modification of the above proof will show $\angle BAO = BCO$, and the four circles are equal.

(II.) Soient $AB = a$, $BC = b$, $OA = \alpha$; $OB = \beta$; $OC = \gamma$; $OD = \delta$.

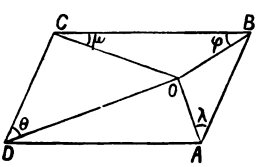
Si R_1, R_2, R_3, R_4 sont les rayons des quatre cercles OAB, OBC, OCD, ODA, on a, d'après les notations de la figure,

$$2R_1 = \frac{\beta}{\sin\lambda} = \frac{\alpha}{\sin(A+\phi)} = \frac{a}{\sin(A+\phi-\lambda)};$$

$$2R_2 = \frac{\gamma}{\sin\phi} = \frac{\beta}{\sin\mu} = \frac{b}{\sin(\mu+\phi)};$$

$$2R_3 = \frac{\delta}{\sin(A-\mu)} = \frac{\gamma}{\sin\theta} = \frac{a}{\sin(A+\theta-\mu)};$$

$$2R_4 = \frac{\alpha}{\sin(A+\theta)} = \frac{\delta}{\sin(A-\lambda)} = \frac{b}{\sin(\theta+\lambda)}.$$



Si $R_1 = R_2$, on a donc $\lambda = \mu$ [l'hypothèse $\lambda + \mu = \pi$ entraînerait en

Digitized by Google

effet cette conséquence, que les deux cercles OAB, OBC seraient confondus en un seul, et non distincts, comme on les suppose], d'où $R_3 = R_4$.

En outre, $$\frac{\alpha}{\gamma} = \frac{\sin(A+\phi)}{\sin\phi} = \frac{\sin(A+\theta)}{\sin\theta},$$

d'où en général $\theta = \phi$, et par suite $R_2 = R_3$.

14988. (The late Professor CAYLEY, F.R.S.)—If

$$l^2+m^2+n^2+\ldots, \quad l'^2+m'^2+n'^2+\ldots$$

are each $= 0$, and s, s' denote positive integers, show that the integral

$$\int (lx+my+nz+\ldots)^s (l'x+m'y+n'z+\ldots)^{s'}\, dx\,dy\,dz\ldots,$$

over the interior of the hyper-sphere $x^2+y^2+z^2+\ldots = 1$ is $= 0$ or $=$ a numerical multiple of $(ll'+mm'+nn'+\ldots)^s$, according as s' is not, or is, $= s$.

Solution by G. H. HARDY, B.A.

It is convenient to write $x_1, x_2, \ldots, l_1, l_2, \ldots, l'_1, l'_2, \ldots$ for $x, y, \ldots, l, m, \ldots, l', m', \ldots$; Λ for $l_1x_1+l_2x_2+\ldots$; and Λ' for $l'_1x_1+l'_2x_2+\ldots$. And we suppose that n is the number of variables. Now

$$\int \Lambda^s \Lambda'^{s'}\, dv = \frac{1}{s+s'+n} \int \Lambda_s \Lambda'^{s'}\, dS,$$

the latter integral being extended over the surface of the unit sphere. Also

$$\Lambda^s \Lambda'^{s'} = (l_1x_1+l_2x_2+\ldots)\,\Lambda^{s-1}\Lambda'^{s'},$$

and, as $x_1, x_2, \ldots$ are the direction cosines of the outward normal at $(x_1, x_2, \ldots)$,

$$\int \Lambda^s \Lambda'^{s'}\, dS = \int \left(\frac{\partial}{\partial x_1}(l_1\Lambda^{s-1}\Lambda'^{s'}) + \frac{\partial}{\partial x_2}(l_2\Lambda^{s-1}\Lambda'^{s'}) \ldots \right) dv.$$

But $$\Sigma\, \partial/\partial x_i\,(l_i\Lambda^{s-1}\Lambda'^{s'}) = (s-1)\,\Lambda^{s-2}\Lambda'^{s'}\,\Sigma\, l_i^2 + s'\Lambda^{s-1}\Lambda'^{s'-1}\,\Sigma\, l_i\, l'_i\,;$$

so that $$\int \Lambda^s \Lambda'^{s'}\, dv = \frac{s'}{s+s'+n}\,\Sigma\, l_i l' \int \Lambda^{s-1}\Lambda'^{s'-1}\, dv.$$

The same reasoning shows that it is also equal to

$$\frac{s}{s+s'+n}\,\Sigma\, l_i l'_i \int \Lambda^{s-1}\Lambda'^{s'-1}\, dv\,;$$

it is therefore equal to 0 unless $s = s'$. And, if $s = s'$,

$$\int \Lambda^s \Lambda'^s\, dv = \frac{s}{2s+n}\,\Sigma\, l_i l'_i \int \Lambda^{s-1}\Lambda'^{s-1}\, dv = \frac{s}{2s+n}\,\frac{s-1}{2s+n-2}\cdots\frac{1}{n+2}\,(\Sigma\, l_i l'_i)^s \int dv$$

$$= \frac{s\,(s-1)\ldots 1}{(2s+n)(2s+n-2)\ldots(n+2)}\,\frac{[\Gamma(\frac{1}{2})]^n}{\Gamma(\frac{1}{2}n+1)}\,(\Sigma\, l_i l'_i)^s$$

$$= \frac{\pi^{\frac{1}{2}n}}{2^s}\,\frac{\Gamma(s+1)}{\Gamma(s+1+\frac{1}{2}n)}\,(\Sigma\, l_i l'_i)^s.$$

This result may easily be verified for small values of n. If $n = 2$, we obtain $$\int (lx+my)^s\,(l'x+m'y)^{s'}\, dv = 0$$

Digitized by Google

or $$= \frac{\pi}{2^s(s+1)}(ll'+mm')^s.$$

The volume integral is

$$\frac{1}{s+s'+2}\int (lx+my)^s(l'x+m'y)^{s'}dS = \frac{l^s l'^{s'}}{s+s'+2}\int_0^{2\pi} e^{\pm is\theta \pm is'\theta}\,d\theta$$

(since $m = \pm il$, $m' = \pm il'$). This is 0 unless $s = s'$ and the signs are different. In this case it is

$$\frac{2\pi(ll')^s}{2(s+1)} = \frac{\pi(ll'+mm')^s}{2^s(s+1)}.$$

If the signs are the same, $ll'+mm' = 0$; so that the formula still holds.

15199. (Professor Neuberg.) — Trouver le lieu du centre d'une quadrique passant par quatre points fixes et dont les axes sont parallèles à des droites données.

Solutions (I.) *by the* Proposer; (II.) *by the* Rev. J. Cullen.

(I.) Soient $F = 0$, $G = 0$, $H = 0$ les équations de trois quadriques de la question. Une quadrique quelconque satisfaisant aux conditions données a pour équation $F + \lambda G + \mu H = 0$, où λ, μ sont des coefficients indéterminés. Le centre étant donné par

$$F'_x + \lambda G'_x + \mu H'_x = 0, \quad F'_y + \lambda G'_y + \mu H'_y = 0, \quad \ldots,$$

le lieu du centre est représenté par

$$\begin{vmatrix} F'_x & G'_x & H'_x \\ F'_y & G'_y & H'_y \\ F'_z & G'_z & H'_z \end{vmatrix} = 0.$$

(II.) Taking the coordinate axes to have the same direction as the given lines, then the equation of any of the quadrics is of the form

$$ax^2 + by^2 + cz^2 + 2px + 2qy + 2rz + 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The centre is given by

$$ax + p = 0, \quad by + q = 0, \quad cz + r = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Now substituting the coordinates (x_1, y_1, z_1), &c. of the four given points in (1), we obtain four equations, and these together with the equations (2) enable us to eliminate a, b, c, p, q, and r, the result being

$$\begin{vmatrix} x, & 0, & 0, & 1, & 0, & 0, & 0 \\ 0, & y, & 0, & 0, & 1, & 0, & 0 \\ 0, & 0, & z, & 0, & 0, & 1, & 0 \\ x_1^2, & y_1^2, & z_1^2, & 2x_1, & 2y_1, & 2z_1, & 1 \\ x_2^2, & y_2^2, & z_2^2, & 2x_2, & 2y_2, & 2z_2, & 1 \\ x_3^2, & y_3^2, & z_3^2, & 2x_3, & 2y_3, & 2z_3, & 1 \\ x_4^2, & y_4^2, & z_4^2, & 2x_4, & 2y_4, & 2z_4, & 1 \end{vmatrix} = 0,$$

which shows that the locus of the centre is a cubic surface of the form

$$Axyz + Byz + Cxz + Dxy + Gx + Fy + Hz + K = 0.$$

Digitized by Google

14881 (Corrected form). (V. DANIEL, B.Sc.)—Determine in Cartesian co-ordinates the family of plane curves related to two focal points P and Q, such that a particle projected through P shall, after rebounding from any point on one such curve, proceed in the line joining Q to that point. Coefficient of rebound $= e$; $PQ = 2a$. Taking for origin the middle point of PQ, show that the lines $x \pm y\sqrt{e} = d$ are common tangents at the focus Q.

Solution by the PROPOSER.

If β and β' are the angles of incidence and reflection, it follows geometrically that

$$\tan\beta = \frac{dx/dy + y/(x+a)}{1 - y/(x+a)\,dx/dy}, \quad \tan\beta' = \frac{dy/dx - (a-x)/y}{1 + (a-x)/y\,dy/dx}.$$

Then $$e = \frac{\tan\beta}{\tan\beta'} = \frac{y^2 - x^2 + a^2 - y(x-a)\,dy/dx + y(x+a)\,dx/dy}{-y^2 + x^2 - a^2 - y(x-a)\,dx/dy + y(x+a)\,dy/dx};$$

therefore $$\frac{1-e}{1+e} \equiv \sin\alpha = \frac{x^2 - a^2 - y^2 + xy\,(dy/dx - dx/dy)}{ay\left\{dy/dx + dx/dy\right\}};$$

therefore $$y^2(dy/dx)^2(x - a\sin\alpha) + y\,dy/dx\,(x^2 - a^2 - y^2) - y^2(x + a\sin\alpha) = 0.$$

Writing $y^2 = Q$ and solving for dQ/dx,

$$(x - a\sin\alpha)\,dQ/dx = Q + a^2 - x^2 \pm \left\{(Q + a^2 - x^2)^2 + 4Q(x^2 - a^2\sin^2\alpha)\right\}^{\frac{1}{2}},$$

an equation which may be transformed thus :—

$$(x - a\sin\alpha)\,dQ/dx + 2(x^2 - a^2\sin^2\alpha)$$
$$= Q + x^2 + a^2\cos 2\alpha \pm \left\{(Q + x^2 + a^2\cos 2\alpha)^2 - 4a^2\cos^2\alpha\,(x^2 - a^2\sin^2\alpha)\right\}^{\frac{1}{2}}.$$

Let $$Q + x^2 + a^2\cos 2\alpha = a\cos\alpha\,(x^2 - a^2\sin^2\alpha)^{\frac{1}{2}}\left\{\theta + \theta^{-1}\right\} \quad \ldots\ldots\ldots\ldots(1).$$

Then, if $x - a\sin\alpha \neq 0$, we obtain the linear equations

$$\frac{dQ}{dx} + 2(x + a\sin\alpha) = 2a\cos\alpha\left\{\frac{x + a\sin\alpha}{x - a\sin\alpha}\right\}^{\frac{1}{2}}\theta,$$

or $$2a\cos\alpha\left\{\frac{x + a\sin\alpha}{x - a\sin\alpha}\right\}^{\frac{1}{2}}\theta^{-1}.$$

Also, differentiating (1),

$$dQ/dx + 2x = a\cos\alpha\,.\,d/dx\left\{(x^2 - a^2\sin^2\alpha)^{\frac{1}{2}}(\theta + \theta^{-1})\right\};$$

therefore $$2\tan\alpha + \frac{d}{dx}\left\{(x^2 - a^2\sin^2\alpha)^{\frac{1}{2}}(\theta + \theta^{-1})\right\} = 2\left\{\frac{x + a\sin\alpha}{x - a\sin\alpha}\right\}^{\frac{1}{2}}\theta$$

or $$2\left\{\frac{x + a\sin\alpha}{x - a\sin\alpha}\right\}\theta^{-1}.$$

Writing $2k_1$ and $2k_2$ for the right-hand expressions gives one and the same differential equation

$$2\tan\alpha + d/dx\left\{k(x - a\sin\alpha) + k^{-1}(x + a\sin\alpha)\right\} = 2k;$$

i.e., $dx/dk - x/\left\{k - 2\tan\alpha/(1 - k^{-2})\right\} = -a\sin\alpha\,(1 + k^{-2})/(k - k^{-1} - 2\tan\alpha)$.

Its solution is

$$x\epsilon^{-H} = -a\sin\alpha\int\epsilon^{-H}\frac{1 + k^{-2}}{k - k^{-1} - 2\tan\alpha}\,dk + C_1 \quad \ldots\ldots\ldots\ldots(2),$$

where $$H = \int\frac{dk}{k - 2\tan\alpha/(1 - k^{-2})};$$

$$\epsilon^{-H} = k^{-1}\left\{\frac{\sec\alpha - \tan\alpha - k^{-1}}{\sec\alpha + \tan\alpha + k^{-1}}\right\}^{-\sin\alpha}(\sec\alpha + \tan\alpha)^{-\sin\alpha}.$$

Digitized by Google

Writing (2) thus, $xe^{-H} = I \cdot a \sin\alpha\,(\sec\alpha+\tan\alpha)^{-\sin\alpha} + C_1$, we have

$$I = -\int \frac{1+k^{-2}}{k-2\tan\alpha-k^{-1}} \left\{\frac{\sec\alpha-\tan\alpha-k^{-1}}{\sec\alpha+\tan\alpha+k^{-1}}\right\}^{-\sin\alpha} k^{-1}\,dk$$

$$= \int \frac{1+k^{-2}}{1-2k^{-1}\tan\alpha-k^{-2}} \left\{\frac{\sec\alpha-\tan\alpha-k^{-1}}{\sec\alpha+\tan\alpha+k^{-1}}\right\}^{-\sin\alpha} dk^{-1};$$

or, since $(1-2\tan\alpha\,k^{-1}-k^{-2}) = \sec^2\alpha\left\{1-(\sin\alpha+k^{-1}\cos\alpha)^2\right\}$

$$\equiv \sec^2\alpha\,(1-K^2), \text{ say,}$$

$$I = \sec\alpha \int \left(\frac{1-K}{1+K}\right)^{-\sin\alpha} \left\{-1+\frac{1-\sin\alpha}{1-K}+\frac{1+\sin\alpha}{1+K}\right\} dK.$$

Now

$$\int \left(\frac{1-K}{1+K}\right)^{-\sin\alpha} dK = K\left(\frac{1-K}{1+K}\right)^{-\sin\alpha} - \int \frac{2K\sin\alpha}{(1+K)^2}\left(\frac{1-K}{1+K}\right)^{-1-\sin\alpha} dK$$

$$= K\left(\frac{1-K}{1+K}\right)^{-\sin\alpha} + \int \left(\frac{1-K}{1+K}\right)^{-\sin\alpha} \times \left\{\frac{1-\sin\alpha}{1-K}+\frac{1+\sin\alpha}{1+K}-\frac{2}{1-K^2}\right\} dK.$$

Hence $\int \left(\frac{1-K}{1+K}\right)^{-\sin\alpha} \left\{-1+\frac{1-\sin\alpha}{1-K}+\frac{1+\sin\alpha}{1+K}\right\} dK$

$$= -K\left(\frac{1-K}{1+K}\right)^{-\sin\alpha} + \int 2\left(\frac{1-K}{1+K}\right)^{-\sin\alpha} \Big/ (1-K^2)\,dK,$$

i.e.,

$$I\cos\alpha = \left(\frac{1-K}{1+K}\right)^{-\sin\alpha} \{\operatorname{cosec}\alpha - K\}.$$

Equation (2) may now be written, after dividing by a constant,

$$x\,(K-\sin\alpha)+a\,(K\sin\alpha-1) = C_2\left(\frac{1-K}{1+K}\right)^{\sin\alpha},$$

or, resubstituting the value $K = \sin\alpha + k^{-1}\cos\alpha$,

$$(x\sec\alpha+a\tan\alpha)\,k^{-1} = a\left\{1+\lambda\left(\frac{\sec\alpha-\tan\alpha-k^{-1}}{\sec\alpha+\tan\alpha+k^{-1}}\right)^{\sin\alpha}\right\} \quad ...(3),$$

where λ is an arbitrary constant.

Equation (1), on substituting

$$\theta = k\left\{\frac{x-a\sin\alpha}{x+a\sin\alpha}\right\}^{\frac{1}{2}},$$

becomes $x^2+y^2+a^2\cos 2\alpha = ka\cos\alpha\,(x-a\sin\alpha)+k^{-1}a\cos\alpha\,(x+a\sin\alpha)$,

or $-y^2\sec^2\alpha = \left\{k\,(x\sec\alpha-a\tan\alpha)-a\right\}\left\{k^{-1}\,(x\sec\alpha+a\tan\alpha)-a\right\}$...(4).

Combining this with (3), we obtain an equation of similar form with (3), and the k-eliminant of these gives the family sought. Thus

$$(x\sec\alpha+a\tan\alpha)\,k^{-1} = a+a\lambda\left\{\frac{\sec\alpha-\tan\alpha-k^{-1}}{\sec\alpha+\tan\alpha+k^{-1}}\right\}^{\sin\alpha},$$

$$(x\sec\alpha-a\tan\alpha)\,k \quad = a-y^2a^{-1}\lambda^{-1}\left\{\frac{\sec\alpha+\tan\alpha+k^{-1}}{\sec\alpha-\tan\alpha-k^{-1}}\right\}^{\sin\alpha}\sec^2\alpha.$$

Multiplying these equations together, and putting $k^{-1}=0$, in which case the first equation gives $x=\infty$, we obtain the asymptotic conics

$$(x^2\sec^2\alpha-a^2\tan^2\alpha) = a^2\,(1+\lambda e^{\sin\alpha})(1-y^2a^{-2}\lambda^{-1}e^{-\sin\alpha}\sec^2\alpha)$$

i.e.,

$$x^2+y^2(1+\lambda^{-1}e^{-\sin\alpha}) = a^2(1+\lambda\,e^{\sin\alpha}\cos^2\alpha).$$

λ is an essentially negative quantity.

Digitized by Google

That may be shown more rigorously by a method which also determines the limit of λ for real members of the family.

Equation (4) may be put into the form

$$y^2\sec^2\alpha+\left\{x\sec\alpha-\tfrac{1}{2}a\,(k+k^{-1})\right\}^2=\left\{\tfrac{1}{2}\,(k-k^{-1})-\tan\alpha\right\}^2a^2\equiv p^2a^2$$

or
$$y^2\sec^2\alpha+\left\{pa+(x\sec\alpha+a\tan\alpha-ak)\right\}^2=p^2a^2,$$

i.e.,
$$\frac{y^2\sec^2\alpha}{p^2}+\left\{1+k\lambda p^{-1}\left(\frac{\sec\alpha-\tan\alpha-k^{-1}}{\sec\alpha+\tan\alpha+k^{-1}}\right)^{\sin\alpha}\right\}^2a^2=a^2,$$

by equation (3), or, since

$$pk^{-1}=\tfrac{1}{2}\,(1-2k^{-1}\tan\alpha-k^{-2})=\tfrac{1}{2}\,(\sec\alpha-\tan\alpha-k^{-1})(\sec\alpha+\tan\alpha+k^{-1})$$
$$\equiv\tfrac{1}{2}h^{-1}l^{-1},\text{ say,}$$
$$y^2\sec^2\alpha/p^2+\left\{1+2\lambda\,h^{1-\sin\alpha}l^{1+\sin\alpha}\right\}^2a^2=a^2\quad\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

λ must evidently be negative when $\sin\alpha$, *i.e.*, $(1-e)/(1+e)$, is irrational to give y real. If e be rational and $=p/q$,

$$1-\sin\alpha\equiv 2p/(p+q)\quad\text{and}\quad 1+\sin\alpha\equiv 2q/(p+q).$$

If $p+q$ be odd, both $h^{1-\sin\alpha}$ and $l^{1+\sin\alpha}$ are positive, and λ must be negative.

If $p+q$ be even and $=2r$, $1-\sin\alpha=p/r$, $1+\sin\alpha=q/r$; so that when p and q are both odd the product $h^{1-\sin\alpha}l^{1+\sin\alpha}$ may be negative. Consequently λ can only be positive when e is the ratio of two odd integers.

The numerical limit of λ is fixed by the maximum value of the product $h^{-1+\sin\alpha}l^{-1-\sin\alpha}$, which will be found to be $e^{-\sin\alpha}=e^{-(1-e)/(1+e)}$. In actually plotting such members of the family in which

$$-\lambda e^{\sin\alpha}=\operatorname*{Lt}_{n=\infty}(1-1/n),$$

k^{-1} will only vary between vanishingly small limits on either side of zero, for we have

$$\operatorname*{Lt}\left\{-\lambda e^{\sin\alpha}(1+e^{\frac{1}{2}}k^{-1})^{-1-\sin\alpha}(1-e^{-\frac{1}{2}}k^{-1})^{-1+\sin\alpha}\right\}=+1,\text{ by (5),}$$

or
$$\log\left\{-\lambda e^{\sin\alpha}\right\}=\log(1+e^{\frac{1}{2}}k^{-1})^{e^{-\frac{1}{2}}\cos\alpha}+\log(1-e^{-\frac{1}{2}}k^{-1})^{e^{\frac{1}{2}}\cos\alpha}$$
$$=\cos\alpha\left\{(k^{-1}-\tfrac{1}{2}e^{\frac{1}{2}}k^{-2}+\ldots)-(k^{-1}+\tfrac{1}{2}e^{-\frac{1}{2}}k^{-2}+\ldots)\right\}$$
$$=-k^{-2}+(\text{higher powers of }k^{-1})$$
$$=\log(1-k^{-2}),\text{ neglecting powers above }k^{-2},$$

i.e., $-\lambda e^{\sin\alpha}\equiv(1-1/n)=1-k^{-2}$, giving $\pm k=\sqrt{n}$ in limit. In fact the asymptotic hyperbola becomes

$$x^2+y^2\left\{1-(1+1/n)\right\}=a^2\left\{1-\cos^2\alpha\,(1-1/n)\right.$$

or
$$x^2-y^2/n=a^2\,(\sin^2\alpha+\cos^2\alpha/n),$$

approaching the geometrically evident solution $x=a\sin\alpha$.

[The Proposer wishes the second part of the Question to be withdrawn.]

14144. (D. Biddle.)—The sum to infinity of the reciprocals of triangular numbers being 2, find the value of a constant, x, to be added to each triangular, such that the sum of the reciprocals shall be $4\pi-1/x$.

Digitized by Google

Solution by Professor SANJÁNA, M.A.

We have $\sum_1^\infty \frac{1}{\frac{1}{2}n(n+1)+x} = 4\pi - \frac{1}{x}$, *i.e.*, $\sum_0^\infty \frac{1}{n^2+n+2x} = 2\pi$.

Now
$$\sum_0^\infty \frac{1}{n^2+n+2x} = \frac{1}{2\iota y}\sum_0^\infty \left\{\frac{1}{n+\frac{1}{2}-\iota y} - \frac{1}{n+\frac{1}{2}+\iota y}\right\}$$
$$= -\frac{\pi}{2\iota y}\sum_0^\infty \left\{\frac{1}{\iota y\pi+(2n+1)\frac{1}{2}\pi} + \frac{1}{\iota y\pi-(2n+1)\frac{1}{2}\pi}\right\}$$
$$= -\frac{\pi}{2\iota y}\tan \iota y\pi = \frac{\pi}{2y}(-\tanh \pi y),$$

where $y = \frac{1}{2}\sqrt{(8x-1)}$. Thus x is found from

$$\tanh[\tfrac{1}{2}\pi\sqrt{(8x-1)}] = -2\sqrt{(8x-1)},$$

which may be solved by means of tables of the hyperbolic functions.

[The PROPOSER states that $x = \frac{3}{32}$, as shown in Appendix I., Vol. L., p. 137.]

15131. (C. E. YOUNGMAN, M.A.)—Given a semicircle on diameter BC, draw an ordinate PN, and on NP produced take A such that $BA^2 = BC(BN + NP)$. Prove that the triangle ABC has

$$\tan^2 A = \tan B \tan C;$$

BROCARD angle $90° - A$; BROCARD axis parallel to BC, &c.

Solutions (I.) *by* R. TUCKER, M.A.; (II.) *by* Professor SANJÁNA, M.A.; (III.) *by* D. BIDDLE.

(I.) Let O be the centre, ON, PN = x, y; radius = a, and AN = Y. Then, from the data,

$$y^2 = a^2 - x^2, \quad (a+x)^2 + Y^2 = 2a(a+x+y), \quad \textit{i.e.}, \quad Y^2 = y^2 + 2ay \ldots \text{(i., ii.)}.$$

Now
$$\tan A = -\tan(B+C) = 2aY/(Y^2 - y^2) = Y/y \ldots\ldots\ldots\ldots \text{(iii.)},$$

and
$$\tan B \tan C = Y^2/(BN.NC) = Y^2/y^2 = \tan^2 A,$$
$$\cot \omega' = \cot B + \cot C + \cot A = (2a+y)/Y = Y/y = \tan A;$$

therefore
$$\omega' = 90° - A.$$

Again, LEMOINE point perpendicular
$$= \tfrac{1}{2}a \tan \omega' = R \sin A \cot A = R \cos A;$$

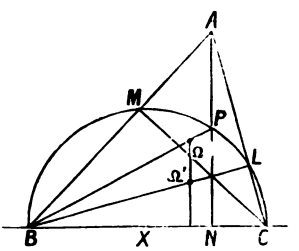
hence BROCARD line is parallel to BC.

(II.) We have

$$BC.NP = BA^2 - BC.BN$$
$$= BA^2 - BP^2 = AN^2 - NP^2;$$

hence

$$\tan A = \frac{\tan BAN + \tan CAN}{1 - \tan BAN \tan CAN}$$
$$= \frac{BC + NA}{1 - BN.NC + AN^2}$$
$$= \frac{BC.NA}{AN^2 - NP^2} = \frac{NA}{NP}.$$



Thus
$$\tan^2 A = NA^2/(BN.NC) = \tan B \tan C.$$

Digitized by Google

Therefore also $\tan^3 A = \tan A \tan B \tan C = \tan A + \tan B + \tan C$;

therefore $\tan A = \cot A + (\tan B + \tan C)/\tan^2 A$

$= \cot A + \cot B + \cot C = \cot \omega$;

so that $\omega = 90° - A$. The perpendicular from circum-centre (O) on BC is $R \cos A$; that from symmedian point (K) is

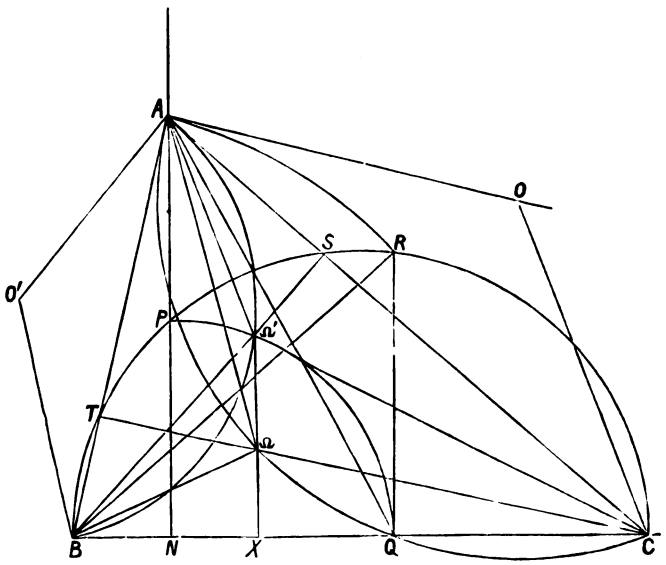
$$\tfrac{1}{2}a \tan \omega = R \sin A \cot A = R \cos A;$$

so that OK is parallel to BC. As $\Omega\Omega'$ is perpendicular to OK, it is perpendicular to BC also; Ω will lie in CM, the perpendicular on AB, and Ω' in BL, that on AC. We shall also have AX, the median, bisecting OK; $\angle A$ greater than 60°, and $\tan A/\tan C = \tan B/\tan A = b^2/c^2$.

(III. (1) Taking BC = unity, we have $BN + NP = BQ = BR^2 = BA^2$; therefore BQ : BA = BA : BC, and the triangles ABC, QBA are similar,

also $\tan^2 A = AN^2/NP^2 = AN^2/(BP^2 . CP^2) = AN^2/(BN . CN)$

$= \tan B \tan C$.



(2) $\cot \omega = \cot A + \cot B + \cot C = NP/AN + BN/AN + CN/AN$

$= (1 + NP)/AN$.

But $AN^2 = AB^2 - BN^2 = BN + NP - BN^2 = NP + NP^2 = (1 + NP)\,NP$;

therefore $AN : NP = 1 + NP : AN$,

and $AN/NP = (1 + NP)/AN = \cot \omega = \tan A = \cot(\tfrac{1}{2}\pi - A)$;

therefore $\omega = \tfrac{1}{2}\pi - A$.

Digitized by Google

(3) Let the foot of the perpendicular from Ω on BC be X, and from Ω' be X'; then we have

$BX' \tan(B-\omega) = (1-BX')\tan\omega$ and $CX \tan(C-\omega) = (1-CX)\tan\omega$;

$\therefore$ $BX' = \sin\omega\cos\omega\cot B + \sin^2\omega$, $CX = \sin\omega\cos\omega\cot C + \sin^2\omega$,

and $BX' + CX = \sin\omega\cos\omega(\cot B + \cot C) + 2\sin^2\omega$

$= (AN \,.\, NP)/(AN^2 + NP^2)\; 1/AN + 2NP^2/(AN^2 + NP^2)$,

But $NP + NP^2 = AN^2$; therefore $BX' + CX = 1 = BC$,

and X' is identical with X. Hence, &c.

Geometrical Solution of Question 14961. See *Ed. Times*, Sept. 1902, and *Reprint* (New Series, Vol. III., p. 62).

By R. F. Davis, M.A.

A simple extension of the theorem given in the *Reprint*, Vol. XXXIV., p. 38, is as follows:—Let B be a fixed point on a given circle, and P a variable point on the same circle. Upon any fixed line through B take points Q, Q' such that BP = BQ = BQ'. Then the envelope of PQ, PQ' is a tricusp (three-cusped hypocycloid) concentric with and circumscribing the given circle.

Now let ABC be a given triangle and E a point on its circum-circle such that BE is parallel to CA. If P be a variable point on the circum-circle and Q, Q' be taken on BE such that BP = BQ = BQ', then PQ, PQ', being equally inclined to CA, BP, are the diameters through P of the two parabolas that can be drawn through ABCP. These envelop a tricusp concentric with and circumscribing the circum-circle of ABC.

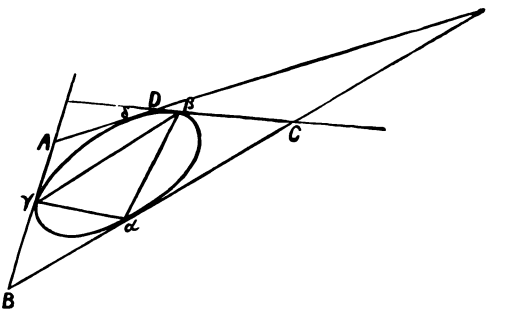
Since the perpendicular from the centroid G of ABC upon the axis of either parabola is one-third the perpendicular from P upon the same line, the envelope of the axis is another tricusp of one-fourth the linear dimensions of the former whose centre is on GO at a distance $= \frac{1}{4}GO$ from G, and circumscribing a circle whose radius $= \frac{1}{4}R$.

15094. (Professor Umes Chandra Ghosh.)—If tangents be drawn to Kiepert's hyperbola of a triangle at its vertices and orthocentre, prove that the three diagonals of the quadrilateral formed by these tangents intersect in pairs at the feet of the perpendiculars of the triangle drawn from its vertices to the opposite sides.

Digitized by Google

Solution by A. M. NESBITT, M.A.

If $\alpha\beta\gamma$ be a triangle inscribed in *any* conic, and if δ be *any* fourth point on the conic, we may regard the following as BRIANCHON hexagons, viz., $A\gamma BC\beta D$, $AB\alpha CD\delta$.



Hence AC, $\gamma\beta$, BD are concurrent, as are also AC, BD, $\alpha\delta$, *i.e.*, the diagonals AC, BD of the quadrilateral ABCD meet at the point where $\alpha\delta$ cuts $\beta\gamma$. We have only to take δ the orthocentre and our conic becomes KIEPERT's hyperbola.

15169. (A. M. NESBITT, M.A.)—A man tosses a coin; he scores 1 for each head that turns up and 2 for each tail. Find the chance of his scoring exactly n.

Solution by J. PRESCOTT, B.A.

Let us consider the first n throws. There are plainly 2^n different arrangements of heads and tails in n throws, all of which are equally likely. Now he could score n by getting n heads in succession. Again, he could score n in the first $(n-1)$ throws by getting $(n-2)$ heads and 1 tail; and then the n-th throw might result in either head or tail. Now the number of ways of getting $(n-2)$ heads and 1 tail in $(n-1)$ throws is ${}^{n-1}C_1$. But the n-th throw may then result in two ways. Thus the number of ways of making n throws so as to score n in the first $(n-1)$ throws is $2\,{}^{n-1}C_1$. Similarly, he could score n in the first $(n-2)$ throws by getting $(n-4)$ heads and 2 tails. The number of ways of doing this is ${}^{n-2}C_2$. Then the remaining two throws may be made in 2^2 ways. Thus the number of ways of making n throws so as to score n in the first $(n-2)$ throws is $2^2\,{}^{n-2}C$.

Hence we see that the total number of ways of scoring exactly n in n throws is $1 + 2\,{}^{n-1}C_1 + 2^2\,{}^{n-2}C_2 + 2^3\,{}^{n-3}C_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\alpha)$.

Now these terms are the coefficients of x^n in the expansions of

$$\frac{1}{1-x},\ \frac{2x^2}{(1-x)^2},\ \frac{2^2x^4}{(1-x)^3},\ \frac{2^3x^6}{(1-x)^4},\ \ldots\ldots.$$

Digitized by Google

Therefore the sum of the terms in (a) is equal to the coefficient of x^n in

$$\frac{1}{1-x}+\frac{2x^2}{(1-x)^2}+\frac{2^2x^4}{(1-x)^3}+\frac{2^3x^6}{(1-x)^4}+\ldots,$$

i.e., in $\dfrac{1/(1-x)}{1-2x^2/(1-x)}$, *i.e.*, in $\dfrac{1}{1-x-2x^2}$, *i.e.*, in $\frac{2}{3}\dfrac{1}{1-2x}+\frac{1}{3}\dfrac{1}{1+x}$, and the coefficient of x^n in the last expression is $\frac{2}{3}2^n+(-1)^n\frac{1}{3}$.

Now the total number of different arrangements is 2^n; therefore the chance of scoring n is $\frac{1}{3}+(-1)^n\dfrac{1}{3.2^n}$.

15108. (R. W. D. Christie.)—Prove by two distinct methods the theorem $2^n \equiv 1 \bmod \frac{1}{2}(p-1)$, when p is a prime of form $4m-1$ and n the number of its primitive roots.

Solution by Lt.-Col. Allan Cunningham, R.E.

It is known (see Mathews's *Theory of Numbers*, Art. 19) that, if $\phi(x)$ denote the *totient* of x, then

$$n = \text{number of prime roots of } p = \phi(p-1) = \phi(4m-2)$$
$$= \phi\{2(2m-1)\} = \phi(2m-1) = \phi.\tfrac{1}{2}(p-1).$$

Also (*op. cit.*, Art. 16), if a be prime to x, then $a^{\phi(x)} \equiv 1 \pmod{x}$; so that, writing $x = 2m-1$, then

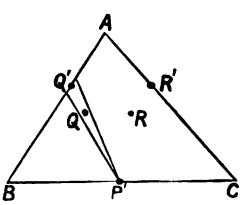
$$a^n = a^{\phi(2m-1)} \equiv 1 \ [\bmod (2m-1)] \equiv 1 \ [\bmod \tfrac{1}{2}(p-1)],$$

and, since 2 is necessarily prime to $(2m-1)$, this is always true when $a=2$.

15145. (R. Chartres.)—Three points are taken at random within a triangle and joined. Find the mean value of the nth power of the area of the triangle thus formed.

Solution by the Proposer *for the case* $n = 1$.

Let P′, Q′, R′ be random points one on each side of the triangle ABC; then evidently the mean value of P′Q′R′ $= \frac{1}{4}\Delta$. Now, if Q move in the element Q′P′, this result will be reduced to $\frac{2}{3}$ of its value; therefore when both Q′ and R′ move in the triangle the M.V. (P′QR) $= \frac{2}{3}\times\frac{1}{4}\Delta = \frac{1}{6}\Delta$.

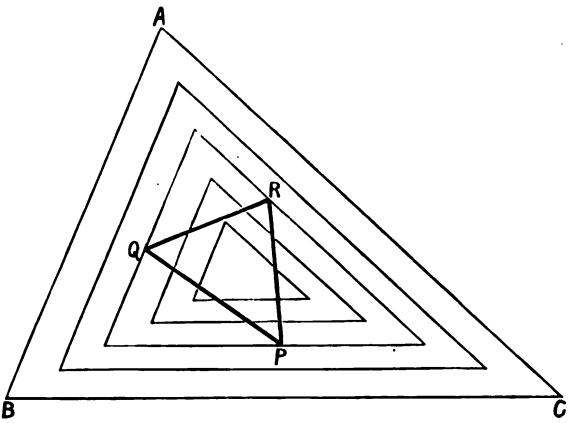


Next, let P′ move parallel to BC, and the M.V. (PQR) $= \frac{3}{4}\times\frac{1}{6}\Delta = \frac{1}{12}\Delta$,

since (1) $\int x\,x^2\,dx \div \int x^2\,dx = \frac{3}{4}$; or (2) by Prebendary Whitworth's theorem M.V. $(x^3) \div$ M.V. $(x^2) = \frac{3}{4}$; or (3) the method is similar to that for finding the distance of the centre of gravity of a cone from the vertex.

Digitized by Google

[Mr. BIDDLE, having seen the PROPOSER's discussion of the above Question, observes regarding the above solution, which deals with the mean value of the triangle formed by joining three points taken at random in a given



triangle, that the result, $\frac{1}{12}\Delta$, agrees with that given in WILLIAMSON's *Integral Calculus* (4th Edition, p. 347), and that the proof, which seems without flaw, is both shorter and simpler. But it is a remarkable fact that, if the points were restricted to the sides of any one (taken at random) of a series of triangles similar to ABC, as represented in the accompanying figure, but supposed infinite in number, the mean value of PQR would be the same.

In the former case, the inverse proportion, as one would suppose, of the frequency of any form of PQR to its size seems to have little or no effect upon the mean value.]

14830. (R. W. D. CHRISTIE.)—Let r be the radius of the base of a segment of a sphere or a paraboloid, and h the height; P, S the solidity of the paraboloid and sphere respectively. Prove $P(3r^2 + h^2) = 3r^2S$, and, if $h = r$, then $4P = 3S$.

Solution by the PROPOSER.

Let $2R$ = diameter of sphere, r = radius of segment's base. Then, by EUCLID III. 30, we have $(2R - h)h = r^2$; therefore

$$\tfrac{1}{6}[\pi h(3r^2 + h^2)] = S = \text{solidity of segment of sphere}$$

and $$\tfrac{1}{2}\pi r^2 h = P = \text{solidity of paraboloid};$$

therefore $P/S = \frac{3}{4}$, if $h = r$.

Generalization of above.—Let $r = h\sqrt{n}$ or $r^2 = h^2 n$; then

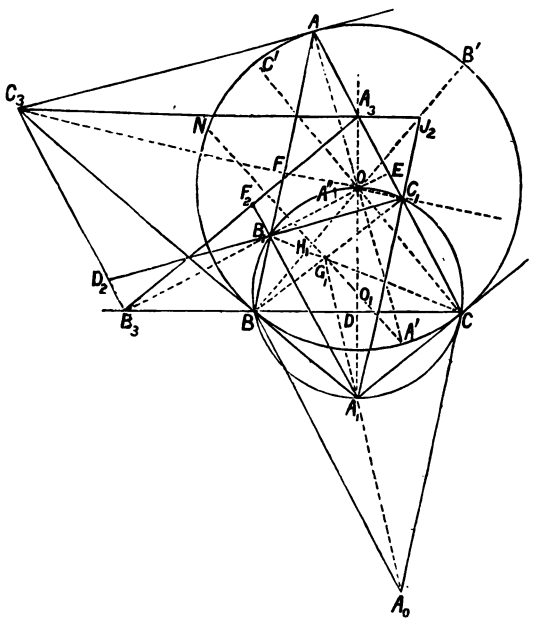
$$P/S = 3n/(2n + 1).$$

Digitized by Google

14885. (Professor SANJÁNA, M.A.)—Let K_1 be the pole of the side BC of a triangle ABC with regard to the circum-circle, and O the circum-centre; let D, E, F be the mid-points of BC, CA, AB respectively; and K_1A_1, K_1B_1, K_1C_1 perpendiculars to OD, OE, OF respectively. Then the six points $K_1 \equiv A_1$, B_1, C_1, B, C, O lie on a circle. So also the similar points $K_2 \equiv B_2$, C_2, A_2, C, A, O and $K_3 \equiv C_3$, A_3, B_3, A, B, O lie on circles. [These circles have been called the ex-Brocardal circles of the triangle.] The triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ are each inversely similar to ABC, and in perspective with it; they are also in perspective two by two, and their axes of perspective meet in a point. [These triangles have been called ex-Brocardal triangles.] The ex-Brocardal triangles form a system of similar figures whose circle of similitude is the circum-circle ABC; whose invariable points are A′, B′, C′ on the circum-circle diametrically opposite to A, B, C, and whose invariable circles are the ex-Brocardal circles (so that every three collinear corresponding points of the figures lie severally on these circles).

Note by C. E. YOUNGMAN, M.A.

These and other properties are proved in the solutions of Questions 14819, &c. (New Series, Vol. I., p. 68, *q.v.*). A few more may be mentioned.



Since A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3 have two by two the same centre of perspective, the three axes will concur. Let $J_2F_2D_2$ be one of

Digitized by Google

them; then angle $B_3D_2B_1 = B = B_3F_2B_1$; therefore B_3D_2, F_2B_1 are concyclic, and $B_3D_2F_1$ is a right angle. Hence $J_1J_2J_3$ is orthologic with ABC.

To follow up the analogy between $A_1B_1C_1$ and the first BROCARD triangle, consider the three similar figures A_1BC, B_1AC, C_1BA on the sides of ABC. The double points are A″, B, C (A″ is the projection of O on AA_1 and lies on the circle $OBCA_1$); A_1, B_1, C_1 are the invariables; and the director-point is A, which is therefore, in any triangle $a_1b_1c_1$ of homologous points, not the centroid (as in the case of the BROCARD triangle), but an ex-centroid, *i.e.*, vertex of parallelogram $Ab_1a_1c_1$. Any triangle *pqr* made by three homologous lines is directly similar to ABC, and its ex-symmedian (homologue of A_1 in ABC) is its centre of perspective with A″BC; in other words, B*q* and C*r* touch the circle *pqr*, and meet on the circle $A''BCOA_1B_1C_1$.

Again, complete the parallelogram $CABA_0$. From Euc. I. 43, *conv.*, it can be shown that A_1A_0, B_1C, C_1B meet at a point G_1; this is the double point of ABC and $A_1B_1C_1$; for the angles G_1BA, G_1CA, $G_1B_1A_1$, $G_1C_1A_1$ all equal A. And G_1 lies at the intersection of the two EULER lines OH and O_1H_1, as may be proved thus: first, A′ lies on O_1H_1; for O is the orthocentre of AC_1B_1, which is equal to $A_1B_1C_1$; therefore $A_1H_1 = AO = OA'$; therefore H_1A' bisects A_1O: and next, G_1 lies on $A'O_1$: for the angle OG_1A' in the figure ABC is equal to O_1G_1O in $A_1B_1C_1$.

Further, the circles ABC, AB_1C_1, BC_1A_1, CA_1B_1 meet at N, the NEUBERG point, on the EULER line of the first BROCARD triangle. N lies also on the EULER lines of $A_1B_1C_1$, $A_2B_2C_2$, and $A_3B_3C_3$. For AN is perpendicular to the EULER line of AC_1B_1; and therefore to that of $A_1B_1C_1$, *i.e.*, to $A'O_1$; and it is also perpendicular to A′N.

The trilinear coordinates of G_1 are given by $a\alpha = (c^2-a^2)\beta/b\,(b^2-a^2)\gamma/c$. It follows that AG_1, BG_2, CG_3 meet BC, CA, AB on the line $\Sigma\,(b^2-c^2)\,\alpha/a = 0$; *i.e.*, on the BROCARD axis, the trilinear polar of N; and A (NBG_1C) is a harmonic pencil.

The line $A_0A_1G_1$ corresponds (in a way) to GK, the "symmedian-point axis," and meets it on BC.

15031. (ALETROP.)—Résoudre en nombres entiers $x^{-2}+y^{-2} = z^{-2}$.

Solution by D. BIDDLE *and* R. CHARTRES.

Take *a*, *b*, *c*, any integral values that will serve to represent the sides of a right-angled triangle, such that $c^2+b^2 = a^2$. Divide both sides of this equation by $a^2b^2c^2$. Then $(ab)^{-2}+(ac)^{-2} = (bc)^{-2}$. The simplest example has $x = 20$, $y = 15$, $z = 12$.

Digitized by Google

The Integration of $\int \frac{d\theta}{\sec\theta + \operatorname{cosec}\theta}$.

By GEORGE SCOTT, M.A.

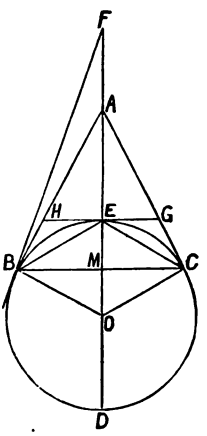
$$\int \frac{d\theta}{\sec\theta + \operatorname{cosec}\theta} = \int \frac{\sin\theta\cos\theta\, d\theta}{\sin\theta + \cos\theta}.$$

Now $2\sin\theta\cos\theta = (\sin\theta + \cos\theta)^2 - 1.$

Therefore
$$\int \frac{d\theta}{\sec\theta + \operatorname{cosec}\theta} = \tfrac{1}{2}\int \frac{d\theta\{(\sin\theta + \cos\theta)^2 - 1\}}{\sin\theta + \cos\theta}$$
$$= \frac{\sin\theta - \cos\theta}{2} - \tfrac{1}{2}\int \frac{d\theta}{\sin\theta + \cos\theta}$$
$$= \frac{\sin\theta - \cos\theta}{2} - \frac{1}{2\sqrt{2}}\int \frac{d\theta}{\sin(\frac{1}{4}\pi + \theta)}$$
$$= \frac{\sin\theta - \cos\theta}{2} - \frac{1}{2\sqrt{2}}\log\tan(\tfrac{1}{8}\pi + \tfrac{1}{2}\theta).$$

Taking Euclid III. 36 *as the Definition of a Circle, show that the ordinary Definition is a Theorem.*

By J. MACLEOD, H.M.I.S.



In the figure the tangents at B, C meet in A. If BO, CO are perpendicular to the respective tangents, BO = CO, and AM bisects BC perpendicularly, all resulting from the evident equality of the tangents.

We can now prove that but one tangent can be drawn at any one point. If possible, at B let there be the two tangents BA, BF.

$AE.AD = AB^2 = AE^2 + EB^2 + 2AE.EM,$

$AE^2 + AE.ED = \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

therefore $AE.ED = EB^2 + 2AE.EM,$

or $AE.MD = EB^2 + AE.EM.$

Hence $MD > EM,$

$FB^2 = FE.FD = (FA + AE)(FA + AD)$

$= FA^2 + FA.AD + FA.AE + AE.AD$

$= FA^2 + AB^2 + \ldots,$

$FA^2 + AB^2 + 2AF.AM = \quad \ldots \quad \ldots \quad \ldots$

Hence $AF.EM = AF.MD,$

which is impossible, and therefore there can be but one tangent at each point.

At E draw the tangent GH, meeting those at C, B in G, H respectively. The following equalities are evident:—BE = EC, GE = GC, HE = HB. Also $\angle$ GCE or GEC = HBE or HEB, and HE = EG and OE is per-

Digitized by Google

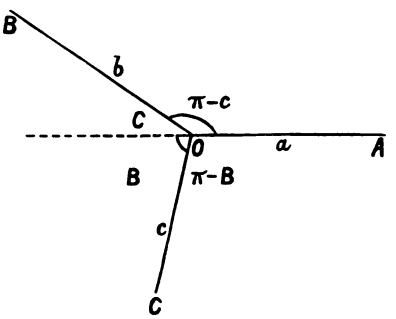
pendicular to HG. Therefore OB, OE, OC are equal. Similarly, it can be proved that all radii within the arc BEC are equal. Again, from the equality of OE to OC, and therefore of the rectangle $AE \cdot AD + OE^2$ to A^2, it follows that ED is bisected in O.

By drawing a tangent at D to meet AC produced it may similarly be proved that all radii within the arc CD are equal to each other, and to those in BEC. Likewise those within BD for similar reasons are equal to each other and to the other radii.

15182. (G. Heppel, M.A.)—The three forces represented by the perpendiculars from the symmedian point to the sides of a triangle are in equilibrium.

Solutions (I.) *by* R. Tucker, M.A.; (II.) *by* Lionel E. Reay, B.A.; (III.) *by* R. F. Davis, M.A., *and* H. Bateman.

(I.) This is equivalent to proving that forces, acting at O, proportional to a, b, c as indicated in the figure equilibrate. Evidently



$$b\cos C + c\cos B = a,$$
$$b\sin C - c\sin B = 0.$$

(II.) If x, y, z are the perpendiculars

$$x/a = y/b = z/c,$$

therefore

$$x/\sin A = y/\sin B = z/\sin C;$$

therefore

$$x/\sin yz = y/\sin zx = z/\sin xy,$$

since $yz = 180° - A$, and similarly; therefore by Lamé's theorem, the forces represented by x, y, z are in equilibrium.

(III.) The symmedian point K of any triangle ABC is the centroid of the triangle formed by joining the feet of the perpendiculars from K on the sides of ABC (Milne's *Companion*, § v., chap. iii., Art. 29).

15146. (Professor Nanson.)—If (A, B, C) is the adjugate of the determinant (a, b, c), prove that

$$(BC, CA, AB) = -(bc, ca, ab)(a, b, c)^2.$$

Digitized by Google

Solutions (I.) *by* T. MUIR, LL.D.; (II.) *by* A. M. NESBITT, M.A.

(I.) The *determinant* (a, b, c) must mean the circulant $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$; which may be denoted by $\circlearrowright | a, b, c |$.

Using thrice the fact that any three-line circulant $\circlearrowright | x, y, z |$ is equal to $(x+y+z)(x^2+y^2+z^2-yz-zx-xy)$, we have two factors on the left, viz., ΣBC and $\Sigma B^2C^2-\Sigma A^2BC$, and six factors on the right, viz., Σbc, $\Sigma b^2c^2-\Sigma a^2bc$, Σa, $\Sigma a^2-\Sigma bc$, Σa, $\Sigma a^2-\Sigma bc$. Now, the first of the former set is resolvable into the first and sixth of the latter set, and the second of the former set is equal to the product of the remaining four of the latter set; in other words, Prof. NANSON's identity implies two, viz.,

$$\frac{\Sigma BC}{\Sigma bc} = \Sigma bc - \Sigma a^2 \quad \text{and} \quad \frac{\Sigma B^2C^2-\Sigma A^2BC}{\Sigma b^2c^2-\Sigma a^2bc} = \Sigma a\, \Sigma a\, (\Sigma a^2-\Sigma bc).$$

(II.) Denote (a, b, c) by Δ. In the determinant

$$\begin{vmatrix} B_1C_1, & B_2C_2, & B_3C_3 \\ C_1A_1, & C_2A_2, & C_3A_3 \\ A_1B_1, & A_2B_2, & A_3B_3 \end{vmatrix}$$

form a new first row by subtracting B_1/A_1 times the second row from the first. This new row will be

$$0, \ \frac{C_2}{A_1}(A_1B_2-A_2B_1), \ \frac{C_3}{A_1}(A_1B_3-A_3B_1), \text{ or } 0, \ C_2\Delta c_3/A_1, \ -C_3\Delta c_2/A_1.$$

Similarly, by subtracting C_1/B_1 times the third row from the second for a new second row, we get $0, -A_2\Delta a_3/B_1, A_3\Delta a_2/B_1$. Thus the proposed determinant

$$= \Delta^2/A_1B_1 \begin{vmatrix} 0, & C_2c_3, & -C_3c_2 \\ 0, & -A_2a_3, & A_3a_2 \\ A_1B_1, & A_2B_2, & A_3B_3 \end{vmatrix} = \Delta^2 \begin{vmatrix} C_2c_3, & -C_3c_2 \\ -A_2a_3, & A_3a_2 \end{vmatrix} = \Delta^2 D, \text{ say.}$$

But, if we treat the determinant $\begin{vmatrix} b_1c_1 \ldots \\ c_1a_1 \ldots \\ a_1b_1 \ldots \end{vmatrix}$ in a similar way, it becomes

$$\begin{vmatrix} 0, & c_2C_3/a_1, & -c_3C_2/a_1 \\ 0, & -a_2A_3/b_1, & a_3A_2/b_1 \\ a_1b_1, & a_2b_2, & a_3b_3 \end{vmatrix}, \ i.e., \ \begin{vmatrix} c_2C_3, & -c_2C_2 \\ -a_2A_3, & a_3A_2 \end{vmatrix} = -D.$$

15109. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Find the least solution in *positive* integers of $x^3+y^3 = mz^3$, where $m = (3\mu-1)$, a positive integer > 2, and $x \neq y$; and x, y, z have no common factor.

Note by R. W. D. CHRISTIE.

(1) If $x^3+y^3 = mz^3$, then

$$\{(x^3+z^3 . m)\, y\}^3 + \{(x^3-y^3)\, z\}^3\, m = \{(y^3+mz^3)\, x\}^3.$$

Digitized by Google

(2) General Solution of Question 15109.—

$$(a^3+3a^2b-b^3)^3+\{3ab(a+b)\}^3=\{a^3+3ab(2a+b)-b^3\}(a^2+ab+b^2)^3$$

for all values of a and b.

$a=2$, $b=1$.—$a=4$, $b=1$ will give solutions of Col. CUNNINGHAM's example, and another.

(3) If $$a^3+b^3=(a+b)z^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

then $$(a+b)^3+(2a-b)^3=9.a.z^3, \quad (a+b)^3+(2b-a)^3=9.b.z^3\ldots(2,3),$$

$$(a-2b)^3+(2a-b)^3=9(a-b)z^3, \quad a^3+(a-b)^3=(2a-b)z^3\ldots\ldots(4,5),$$

$$b^3+(b-a)^3=(2b-a)z^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

(4) Two other general solutions are:—

(a) $$\{a(a^3-2b^3)\}^3+\{b(2a^3-b^3)\}^3=(a^3-b^3)(a^3+b^3)^3,$$

(b) $$(a^4+b^4-2ab^3+a^3b-3a^2b^2)^3+\{3ab(a^2-b^2)\}^3$$
$$=(a^3+3a^2b-6ab^2+b^3)(a^3+b^3)^3.$$

From (b) the common factor $(a+b)^3$ may be eliminated, but it is left thus designedly to show that when 4 (a) is fixed then 4 (b) can be obtained by adding fixed constants to the cubes; *e.g.*, $a=3$, $b=1$ gives, for 4 (a), $75^3+53^3=26.28^3$, and consequently, by adding the constants $b(b^3+a^3-3a^2b)$, $b(a^3+b^3-3ab^2)$, $b(2b^3+3a^2-6ab)$, *i.e.*, 1, 19, 11 respectively, we secure $76^3+72^3=37.28^3$.

15119. (H. BATEMAN,)—ξ, η, ζ are the distances of a point from the angular points of a triangle. Prove that the two points for which $\xi^l\eta^m\zeta^n$ is a maximum are the foci of the ellipse whose areal equation is

$$\surd(lx)+\surd(my)+\surd(nz)=0.$$

Solutions (I.) *by* G. F. S. HILLS; (II.) *by* Professor SANJÁNA, M.A.

(I.) $u=\xi^l\eta^m\zeta^n$ to be made a maximum subject to condition

$$a^2\xi^4\ldots-2bc\cos A\eta^2\zeta^2\ldots-2a^2bc\cos A\xi^2\ldots+a^2b^2c^2=0,$$

which is the relation connecting the distance of four points in a plane (SALMON, *Conics*, p. 134).

Differentiating, &c., we get

$$[a\xi(a\xi^2-b\cos C\eta^2-c\cos B\zeta^2-abc\cos A)]/l/\xi=\ldots=\ldots.$$

Now $$\xi^2=bc\cos A(x-1)^2+ca\cos By^2+ab\cos Cz^2, \ \ldots,$$

formula for distance in areals; therefore

$$a\xi^2-b\cos C\eta^2-c\cos B\zeta^2-abc\cos A=-2abc\sin B\sin Cx,$$

on easy reduction; therefore $\xi^2x/l=\eta^2y/m=\zeta^2z/n$ gives the maximum points, or, substituting value for ξ^2, and putting $1-x=y+z$,

$$[(c^2y^2+b^2z^2-2bc\cos Ayz)x]/l=\ldots=\ldots,$$

which are the foci of $\surd(lx)+\surd(my)+\surd(nz)=0$.

Digitized by Google

[A proof of this latter is, if (x, y, z) be one focus, (x', y', z') the other,

$xx'/a^2 = yy'/b^2 = zz'/c^2 = k^2$, whence $k^2(a^2yz + b^2zx + c^2xy) = xyz$,

and centre is $\frac{1}{2}(x + x') = \frac{1}{2}(x^2 + k^2a^2)/x, \ldots$;

but centre from usual formula is $(m + n, n + l, l + m)$; hence, equating these, we have $\{x(a^2yz + b^2zx + c^2xy) - yz(x + y + z)a^2\}/(m + n) = \ldots = \ldots$,

whence above equations for foci easily follow.]

(II.) If the trilinear co-ordinates of a point be f, g, h, its distances ξ, η, ζ from A, B, C are given by $\xi^2 \sin^2 A = h^2 + g^2 + 2gh \cos A$, and two similar relations. Omitting a constant factor, we therefore have to find the maximum of

$$u \equiv (g^2 + h^2 + 2gh \cos A)^{\frac{1}{2}l} (h^2 + f^2 + 2hf \cos B)^{\frac{1}{2}m} (f^2 + g^2 + 2fg \cos C)^{\frac{1}{2}n},$$

subject to the condition $af + bg + ch = 2\Delta$. Differentiating logarithmically, we have

$$\frac{1}{u} Du = \left\{ \frac{m(f + h \cos B)}{h^2 + f^2 + 2hf \cos B} + \frac{n(f + g \cos C)}{f^2 + g^2 + 2fg \cos C} \right\} Df$$
$$+ \left\{ \frac{l(g + h \cos A)}{g^2 + h^2 + 2gh \cos A} + \frac{n(g + f \cos C)}{f^2 + g^2 + 2fg \cos C} \right\} Dg$$
$$+ \left\{ \frac{l(h + g \cos A)}{g^2 + h^2 + 2gh \cos A} + \frac{m(h + f \cos B)}{h^2 + f^2 + 2hf \cos B} \right\} Dh,$$

which should be zero for a maximum. Also $aDf + bDg + cDh = 0$; hence the co-ordinates of the required point are given by

$$[M(f + h \cos B) + N(f + g \cos C)]/a = [L(g + h \cos A) + N(g + f \cos C)]/b$$
$$= [L(h + g \cos A) + M(h + f \cos B)]/c,$$

where $L \equiv l/(g^2 + h^2 + 2gh \cos A), \ldots$. Eliminating N between these, we get, after some obvious reduction,

$Lcg \sin A(f \sin A + g \sin B + h \sin C) = Mcf \sin B(f \sin A + g \sin B + h \sin C)$,

or $La/f = Mb/g$; so also each of these $= Nc/h$. Thus the trilinear co-ordinates of the point satisfy the relations

$$[f(g^2 + h^2 + 2gh \cos A)]/la = [g(h^2 + f^2 + 2hf \cos B)]/mb$$
$$= [h(f^2 + g^2 + 2fg \cos C)]/nc.$$

Now the trilinear equivalent of $\Sigma\sqrt{(lx)} = 0$ is $\Sigma\sqrt{(laa)} = 0$, and, if f_1, g_1, h_1 be the co-ordinates of the focus of this inscribed conic, its equation is

$$\Sigma\{a(g_1h_1^{-1} + h_1g_1^{-1} + 2 \cos A)\}^{\frac{1}{2}} = 0.$$

[See my Question 14289, August, 1899.] On comparing, we get

$$(g_1h_1^{-1} + h_1g_1^{-1} + 2 \cos A)/la = (h_1f_1^{-1} + f_1h_1^{-1} + 2 \cos B)/mb$$
$$= (f_1g_1^{-1} + g_1f_1^{-1} + 2 \cos C)/nc,$$

i.e., $[f_1(g_1^2 + h_1^2 + 2h_1g_1 \cos A)]/la = [g_1(h_1^2 + f_1^2 + 2h_1f_1 \cos B)]/mb$
$$= [h_1(f_1^2 + g_1^2 + 2f_1g_1 \cos C)]/nc,$$

which shows that (f, g, h) and (f_1, g_1, h_1) are one and the same point.

14918. (Rev. T. Roach, M.A.)—If from S′, the second focus, a line be drawn parallel to SP, meeting the ellipse in P′ and SY in Z, show that SP′Z is an isosceles triangle.

Digitized by Google

Solutions (I.) *by the* PROPOSER; (II.) *by* A. W. POOLE.

(I.) $SZ = S'W$,

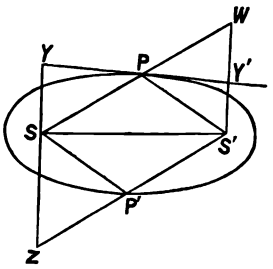
opposite side of parallelogram

$= SP + S'P = AA'$

$= SP' + S'P'$;

therefore $SP' = P'Z$;

therefore triangle is isosceles.

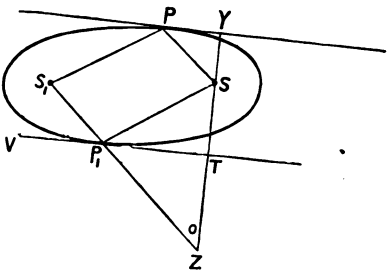


(II.) Tangent at P_1 is parallel to tangent at P; therefore angles at T are right. Also

$\angle S_1P_1V = SP_1T$,

also $= ZP_1T$;

therefore remaining angles P_1ST, P_1ZT are equal; therefore triangle SP_1Z is isosceles.



15172. (Professor NANSON.)—Sum to infinity the series

$$\frac{1}{1.2} - \frac{1}{3.4} + \frac{1}{6.7} - \frac{1}{8.9} + \frac{1}{11.12} - \dots .$$

Solution by Professor SANJÁNA, M.A.

Let $f(x) = x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{6}x^6 - \frac{1}{7}x^7 - \frac{1}{8}x^8 + \frac{1}{9}x^9 + \dots$;

then $f(1)$ = the series given, and $f(0) = 0$. Also

$$\begin{aligned} f'(x) &= 1 - x - x^2 + x^3 + x^5 - x^6 - x^7 + x^8 + x^{10} - x^{11} \dots \\ &= 1 - x(1+x) - x^6(1+x) - x^{11}(1+x) - \dots + x^3(1+x^2) + x^8(1+x^2) + \dots \\ &= 1 - \frac{(1+x)x}{1-x^5} + \frac{(1+x^2)x^3}{1-x^5} = \frac{(1-x)(1-x^2)}{1-x^5} = \frac{1-x^2}{1+x+x^2+x^3+x^4}. \end{aligned}$$

Hence, series $= f(1) - f(0) = \int_0^1 f'(x)\,dx$

$$\begin{aligned} &= \int_0^1 \frac{(1-x^2)\,dx}{1+x+x^2+x^3+x^4} = \frac{1}{\sqrt{5}} \left\{ \log \frac{x^2 + \frac{1}{2}(\sqrt{5}+1)x + 1}{x^2 - \frac{1}{2}(\sqrt{5}-1)x + 1} \right\}_0^1 \\ &= \frac{1}{\sqrt{5}} \log \frac{\sqrt{5}+1}{\sqrt{5}-1} = \frac{1}{\sqrt{5}} \left\{ \log(3+\sqrt{5}) - \log 2 \right\}. \end{aligned}$$

Digitized by Google

15216. (R. C. ARCHIBALD, M.A., Ph.D.)—The *trisectrix* of MACLAURIN ($r\cos\frac{1}{3}\phi = C$ or $r\sin 2\theta + a\sin 3\theta = 0$), the *logocyclic curve* or *strophoide* ($r\cos\theta + a\cos 2\theta = 0$), and the *cubic* of TSCHIRNHAUSEN ($r^{\frac{1}{3}}\cos\frac{1}{3}\theta = a^{\frac{1}{3}}$) are nodal cubics. Express their equations in the form $X^{\frac{1}{3}} + Y^{\frac{1}{3}} + Z^{\frac{1}{3}} = 0$, where X, Y, Z are linear functions of the Cartesian co-ordinates.

Solution by θ. β.

When the equation to a cubic takes the form $(X+Y+Z)^3 = 3\lambda XYZ$, the triangle of reference is formed by three inflexional tangents, $X+Y+Z=0$ being the line on which the three inflexions are situated. Now, in the case of a nodal cubic, there is but one real inflexion (assuming, as is the case here, that the tangents at the node are *real*), and there are two imaginary inflexions; so the reduction in question is unique. The method of obtaining it is now obvious: we must find the three inflexions; then the corresponding tangents from the lines $X=0$, $Y=0$, $Z=0$; the numerical multipliers are to be chosen so that $X+Y+Z=0$ is the line of inflexions.

(i.) $$r\sin 2\theta + a\sin 3\theta = 0.$$

If $\tan\theta = t$, this becomes, in Cartesians,

$$x/(t^2-3) = y/[t(t^2-3)] = a/(t^2+1).$$

In order that a line $lx+my+na=0$ may meet this curve in three consecutive points, we have either

$$(a)\quad m=0,\quad l+n=0,\quad t=\infty,$$

or $$(b)\quad l/1 = m/\pm i = n/2,\quad t=\pm i.$$

Thus the three inflexions are all at infinity; and the three inflexional tangents are

$$X = x+iy+2a=0,\quad Y = x-iy+2a=0,\quad Z = 2a-2x=0,$$

where the constants in Z have been chosen so that $X+Y+Z=0$ is the line at infinity, on which the inflexions are here situated.

Substituting in the equation $y^2(a-x) = x^2(x+3a)$, we find that it reduces to the form $(X+Y+Z)^3 - 27XYZ = 0$, *i.e.*, $X^{\frac{1}{3}} + Y^{\frac{1}{3}} + Z^{\frac{1}{3}} = 0$.

(ii.) $$r\cos\theta + a\cos 2\theta = 0.$$

Here, again, let $t = \tan\theta$, and then we have

$$x/(t^2-1) = y/[t(t^2-1)] = a/(t^2+1)$$

in Cartesians. If $lx+my+na=0$ cuts the curve in three coincident points, we have

$$(a)\quad m=0,\quad l+n=0,\quad t=\infty;$$

$$(b)\quad l/5 = m/(\pm 3i\sqrt{3}) = n/4,\quad t=\pm i/\sqrt{3}.$$

In this case the three inflexions are on the line $x+2a=0$; and the three tangents are

$$X = 5x+3\sqrt{3}\,iy+4a=0,\quad Y = 5x-3\sqrt{3}\,iy+4a=0,\quad Z = 4a-4x=0,$$

the constants in Z being chosen so that $X+Y+Z=0$ is the same line as $x+2a=0$.

Digitized by Google

Substituting in the equation $y^2(a-x) = x^2(a+x)$, we find

$$(X+Y+Z)^3 - 27XYZ = 0,$$

as before.

(iii.) $r^{\frac{1}{3}} \cos \frac{1}{3}\theta = a^{\frac{1}{3}}$, or $x = r\cos\theta = 4a - 3a^{\frac{1}{3}}r^{\frac{2}{3}}$.

Hence we have $(x-4a)^3 + 27ar^2 = 0$.

So here we have the reduction made by writing

$$X = x+iy, \quad Y = x-iy, \quad Z = -8a,$$

for then $X+Y+Z = 2(x-4a), \quad XY = r^2,$

and so $(X+Y+Z)^3 - 27XYZ = 0.$

15086. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Factorize completely (into prime factors) $N = (60{,}000^6 + 1)$.

Solution by the PROPOSER.

Writing $N = x^6 + y^6 = N_1 N_2,$

where $N_1 = x^2 + y^2, \quad N_2 = (x^6 + y^6) \div (x^2 + y^2).$

Making $x = 1$, $y = 6 \cdot 10^4$ gives the proposed N.

As to N_1.—A MS. table, prepared by the PROPOSER, showing solutions (y) of the congruence $y^2 + 1 \equiv 0 \pmod{p}$, gives 13 and 1009 as divisors, when $y = 6 \cdot 10^4$. Hereby $N_1 = 13 \cdot 1009 \cdot 274453$ (all primes).

As to N_2.—Every number N_2 of form $(x^6 + y^6) \div (x^2 + y^2)$ is *algebraically* resolvable into two factors L, M, when $x = \xi^2$, $y = 6\eta^2$; given by

$$L = P - Q, \quad M = P + Q, \quad P = x^2 + 3xy + y^2, \quad Q = 6\xi\eta(x+y).$$

[This resolution is said to be due to the late M. AURIFEUILLE; see ED. LUCAS's paper, *Sur la Série récurrente de Fermat*, Rome, 1879, pp. 5, 6; this function (N_2) has been styled a *Sext-Aurifeuillian* by the PROPOSER.]

Taking $\xi = 1$, $\eta = 100$ gives $x = 1$, $y = 6 \cdot 10^4$ (as required):

$$L = 3564{,}179{,}401; \quad M = 3636{,}180{,}601.$$

A MS. table, prepared by the PROPOSER, showing solutions (y) of $(y^6 + 1) + (y^2 + 1) \equiv 0 \pmod{p}$, gives 37 and 9157 as divisors of N_2, when $y = 6 \cdot 10^4$. On dividing out

$$L = 37 \cdot 96{,}329{,}173; \quad M = 9157 \cdot 397093;$$

and the large factors in L, M must be *prime*, because the MS. table quoted shows that this N_2 has *no more* divisors $< 16{,}000$.

2039. (Professor WOLSTENHOLME.)—Find the mean value of the perimeters of inscribed triangles in a given circle.

10933. (W. J. C. MILLER, B.A.)—If P, Q be two random points inside a circle whose centre is C, find the average of (1) the perimeter, (2) the area, (3) the sum of the squares on the sides of the triangle CPQ.

Digitized by Google

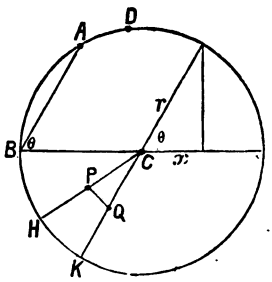
Solution by R. CHARTRES.

(2039). The mean value of the perimeter of the triangle BAD

$$= 3M(AB) = 6M(x)$$
$$= 6(2r/\pi)$$

(Euc. VI. C. extension). The mean value of the sum of the n-th powers of the sides

$$= 3M(AB^n)$$
$$= 3(2r)^n \int_0^{\frac{1}{2}\pi} \cos^n\theta\, d\theta + \tfrac{1}{2}\pi.$$



(10933). (1) The mean value of the perimeter of CPQ

$$= M(CP) + M(CQ) + M(PQ) = \tfrac{4}{3}r + \tfrac{4}{3}r + 128r/45\pi.$$

(2) M (area of triangle CHK) $= \frac{1}{2}r\,M$ (perpendicular) $= \frac{1}{2}r\,2r/\pi = r^2/\pi$ (Euc. VI. C. extension); therefore M (area CPQ) $= (\frac{2}{3})^2 r^2/\pi = 4r^2/9\pi$.

(3) M (sum of the squares of the sides)

$$= M(CP^2) + \ldots = \tfrac{1}{2}r^2 + \tfrac{1}{2}r^2 + r^2 = 2r^2.$$

An exhaustive solution to this Question by quadruple integration is found in *Reprint*, Vol. LV., and the results in (2) and (3) are the same as here given, but the result in (1) was so different from mine that I was induced to go through the integrations, where I found a misprint, and, this being corrected, the result is the same as mine $= \frac{4}{3}r + 128r/45\pi$.

9076. (Professor NASH, M.A.)—If $A\omega$, $B\omega$, $C\omega$, $A\omega'$, $B\omega'$, $C\omega'$ meet the circum-circle in X, Y, Z, X′, Y′, Z′, show that (1) the triangles XYZ, X′Y′Z′ are similar and equal to ABC, and are TUCKER'S triangles of another triangle A′B′C′, similar and similarly situated to ABC, the centre of similitude being the pole of $\omega\omega'$ with respect to the BROCARD circle; and (2) area A′B′C′ : area ABC $= \sin^2 3\omega : \sin^2\omega$.

Solution by the Rev. J. CULLEN.

(1) It is easy to see that the equation of XX′ is

$$\alpha + 2a^3\Delta/\lambda^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i.)},$$

where $\lambda^2 = \Sigma b^2c^2$; hence XX′ (or B′C′) is parallel to BC. Similarly YY′ and ZZ′ are parallel to AC and AB respectively; so that the triangle A′B′C′ is similar to ABC. It is also similarly situated, since from (i.) it follows that the point (a^3, b^3, c^3) is the centre of similitude, which is also the pole of $\Sigma\, a\omega_a/a = 0$ (*i.e.*, $\Omega\Omega'$) with respect to $abc\Sigma a^2 - \Sigma a^3\beta\gamma = 0$, the BROCARD circle.

Now, if we refer $\Sigma\, a/\alpha = 0$ .....(ii.),

Digitized by Google

the circum-circle of ABC, to A′B′C′ as the triangle of reference, (ii.) becomes $\Sigma a/(\alpha-\mu a^2) = 0$ .....(iii.), which can be expressed in the form

$$(\alpha-\sigma a)(\beta-\sigma b)(\gamma-\sigma c)-a\beta\gamma = 0 \text{.....(iv.)},$$

where $\mu = 2\Delta/\lambda^2$, $\sigma = 2\Delta K/(K^2-\lambda^2)$, and $K = \Sigma a^2$. The form of (iv.) shows that (ii.) is a TUCKER circle of A′B′C′, whence XYZ and X′Y′Z′ are TUCKER triangles, which of course are similar to A′B′C′, and therefore also to ABC. Again, they are equal to ABC, since the three triangles have a common circum-circle.

(2) The perpendicular α on BC from the point (a^3, b^3, c^3)

$$= 2a^3\Delta/(K^2-2\lambda^2),$$

while α', that on B′C′, is by (1) $= 2a^3\Delta/\lambda^2 + 2a^3\Delta/(K^2-2\lambda^2)$; whence the modulus of similarity $\alpha'/\alpha = [1/\lambda^2 + 1/(K^2-2\lambda^2)]/[1/(K^2-2\lambda^2)] = K^2/\lambda^2-1$, but $K/\lambda = 2\cos\omega$; therefore $\alpha'/\alpha = \sin 3\omega/\sin\omega$, or area of A′B′C′ : area of ABC :: $\sin^2 3\omega : \sin^2\omega$.

8871. (Professor MAHENDRA NATH RAY, M.A., LL.B.)—If

$$S_r = \frac{1}{2r+1} - \frac{1}{2n+2r+1} + \frac{1}{4n+2r+1} - \frac{1}{6n+2r+1} + \text{... ad inf.},$$

n being a positive integer, show that

$$S_0 - kS_1 + \frac{k(k-1)}{2!}S_2 - \frac{k(k-1)(k-2)}{3!}S_3 + \text{... to } k+1 \text{ terms} = \int_0^\pi \frac{x\sin^{2k+1}x\,dx}{\pi(1+\cos^{2n}x)}.$$

Solution by Professor SANJÂNA, M.A.

Evidently $S_0 = \int_0^1 (1-x^{2n}+x^{4n}-\text{... ad inf.})\,dx = \int_0^1 \frac{dx}{1+x^{2n}}$;

$$S_1 = \int_0^1 \frac{x^2\,dx}{1+x^{2n}};\quad S_2 = \int_0^1 \frac{x^4\,dx}{1+x^{2n}};\quad \text{....}$$

Thus the sinister $= \int_0^1 \frac{(1-x^2)^k\,dx}{1+x^{2n}} = \int_0^{\frac{1}{2}\pi} \frac{\sin^{2k+1}\theta}{1+\cos^{2k}\theta}\,d\theta,$

on putting $x = \cos\theta$ and simplifying. The dexter integral $= I$, say,

$$= \int_0^\pi \frac{(\pi-x)\sin^{2k+1}(\pi-x)}{\pi[1+\cos^{2k}(\pi-x)]}\,dx = \int_0^\pi \frac{\sin^{2k+1}x}{1+\cos^{2k}x}\,dx - I;$$

hence $2I = \int_0^\pi \frac{\sin^{2k+1}\theta}{1+\cos^{2k}\theta}\,d\theta = 2\int_0^{\frac{1}{2}\pi} \frac{\sin^{2k+1}\theta}{1+\cos^{2k}\theta}\,d\theta.$

Hence the result.

Digitized by Google

7296. (T. Muir, M.A., F.R.S.E.)—Show that, if Ω denote the bordered Hessian $\begin{vmatrix} 0 & a & b & c \\ a & P & Q & S \\ b & Q & R & T \\ c & S & T & U \end{vmatrix}$,

(1) the equation $\left(\frac{d}{dT}x - \frac{d\Omega}{dS}y\right) = 4c^2\Omega$ denotes two straight lines passing through the two points in which the conic

$$Px^2 + 2Qxy + Ry^2 + 2Sx + 2Ty + V = 0$$

is cut by the straight line $ax + by + c = 0$; and (2) the equation of the straight line which completes the inscribed quadrilateral of which the said three lines are sides is

$$\begin{vmatrix} 0 & a & b & c \\ a & P & Q & S \\ b & Q & R & T \\ c & S & T & 2cS/a \end{vmatrix} ax + \begin{vmatrix} 0 & a & b & c \\ a & P & Q & S \\ b & Q & R & T \\ c & S & T & 2cT/b \end{vmatrix} by - \begin{vmatrix} 0 & a & b & c \\ a & P & Q & S \\ b & Q & R & T \\ c & S & T & 0 \end{vmatrix} c = 0.$$

Solution by Professor Nanson.

The tangential equation of the points of intersection of the line and conic $\quad U \equiv lx + my + nz = 0, \quad S \equiv (abcfgh)(xyz)^2 = 0$

is $$(ABCFGH)(l'm'n')^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where l', m', n' are current tangential co-ordinates and A, ... are the cofactors of a, ... in the bordered discriminant Ω of S, U. Hence the equation of the joins of the point (x', y', z') to the intersections of U, S is

$$\begin{vmatrix} A & H & G & x' & x \\ H & B & F & y' & y \\ G & F & C & z' & z \\ x' & y' & z' & & \\ x & y & z & & \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Now, taking $x' = G$, $y = F$, $z' = 0$, this reduces to

$$C \begin{vmatrix} G & x \\ F & y \end{vmatrix}^2 = z^2 \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & \end{vmatrix} = -Cz^2(AB - H^2),$$

because the discriminant of (1) vanishes. But $AB - H^2 = -n^2\Omega$, and therefore $(Gx - Fy)^2 = n^2\Omega z^2$, which is the first result to be proved, since

$$A = \tfrac{1}{2}(d\Omega/da), \ \ldots .$$

With the usual notation, the joins (2) are also given by

$$SU'^2 - 2PU'U + S'U^2 = 0,$$

and hence the fourth side of the inscribed quadrilateral is $2PU' - S'U = 0$.
Now, when $x' = G$, $y' = F$, $z' = 0$, we have

$$U' = lG + mF = -nC,$$

$$P = (aG + hF)x + (hG + bF)y + (gG + fF)z = -C(gx + fy + cz) - NU + \Omega x,$$

$$S' = -C(gG + fF) + nNC, = C(2nN + cC - \Omega),$$

and thus the equation of the fourth side is

$$(\Omega - cC)(lx + my - nz) + 2n(gx + fy)C = 0,$$

and this may be written in the determinant form given in the Question.

Digitized by Google

15142. (Professor NANSON.)—If

$$x+y+z+w=0, \quad ax+by+cz+dw=0,$$

$$(a-d)^2(b-c)^2(xw+yz)+(b-d)^2(c-a)^2(yw+zx)+(c-d)^2(a-b)^2(zw+xy)=0,$$

prove that $x:y:z:w=(d-b)(d-c)(b-c):(d-c)(d-a)(c-a):(d-a)(d-b)(a-b):(b-c)(c-a)(a-b)$.

Solution by A. W. POOLE.

$$x+y+z+w=0 \quad \ldots\ldots (1),$$

$$ax+by+cz+dw=0 \quad \ldots\ldots (2),$$

$$\Sigma(a-d)^2(b-c)^2(xw+yz)=0 \quad \ldots\ldots (3).$$

Eliminate w from (1) and (2), and we have $\Sigma(a-d)x=0$.

Put now $a-d=\alpha,\ b-d=\beta,\ c-d=\gamma$;

therefore $b-c=\beta-\gamma, \ldots$;

therefore we have $\alpha x+\beta y+\gamma z=0 \quad \ldots\ldots (5)$

and $\Sigma\alpha^2(\beta-\gamma)^2(xw+yz)=0 \quad \ldots\ldots (4).$

Eliminate w from (1) and (4), and we have

$$\Sigma\alpha^2(\beta-\gamma)^2[x(x+y+z)-yz]=0$$

or $$\Sigma x^2\alpha^2(\beta-\gamma)^2+\Sigma yz\{-\alpha^2(\beta-\gamma)^2+\beta^2(\gamma-\alpha)^2+\gamma^2(\alpha-\beta)^2\}=0.$$

Now
$$\begin{aligned}-\alpha^2(\beta-\gamma)^2+\beta^2(\gamma-\alpha)^2+\gamma^2(\alpha-\beta)^2&=(\beta\gamma-\alpha\gamma)(\beta\gamma-\beta\alpha-\alpha\beta+\alpha\gamma)+\gamma^2(\alpha-\beta)^2\\&=\gamma(\beta-\alpha)(\beta\gamma-2\alpha\beta+\alpha\gamma)+\gamma^2(\alpha-\beta)^2\\&=\gamma(\alpha-\beta)\{-\beta\gamma+2\alpha\beta-\alpha\gamma+\gamma\alpha-\gamma\beta\}\\&=\gamma(\alpha-\beta)(2\alpha\beta-2\beta\gamma)\\&=-2\beta\gamma(\alpha-\beta)(\gamma-\alpha);\end{aligned}$$

therefore our equations are

$$\alpha x+\beta y+\gamma z=0, \quad \Sigma\alpha^2(\beta-\gamma)^2x^2-2\Sigma yz\beta\gamma(\alpha-\beta)(\gamma-\alpha)=0.$$

Eliminate z between $\alpha x+\beta y+\gamma z=0$,

$$l^2x^2+m^2y^2+n^2z^2-2mnyz-2nlzx-2lmxy=0,$$

where $l=\alpha(\beta-\gamma), \ldots.$

We have

$$(l^2x^2+m^2y^2)\gamma^2+n^2(\alpha x+\beta y)^2-2lmxy\gamma^2+2(mny+nlx)\gamma(\alpha x+\beta y)=0$$

or $$x^2(l^2\gamma^2+n^2\alpha^2+2nl\alpha\gamma)+y^2(m^2\gamma^2+\beta^2n^2+2mn\gamma\beta)+2xy(n^2\alpha\beta-lm\gamma^2+nl\gamma\beta+mn\gamma\alpha)=0.$$

Coefficient of $x^2=(l\gamma+n\alpha)^2=[\alpha\gamma(\beta-\gamma)+\gamma(\alpha-\beta)\alpha]^2=\alpha^2\gamma^2(\alpha-\gamma)^2$,

,, $y^2=(m\gamma+n\beta)^2=[\beta\gamma(\gamma-\alpha)+\beta\gamma(\alpha-\beta)]^2=\beta^2\gamma^2(\gamma-\beta)^2$,

,, $$\begin{aligned}2xy&=n^2\alpha\beta-lm\gamma^2+nl\beta\gamma+mn\gamma\alpha\\&=\gamma^2(\alpha-\beta)^2\alpha\beta-\alpha(\beta-\gamma)\beta(\gamma-\alpha)\gamma^2+\gamma(\alpha-\beta)\alpha(\beta-\gamma)\beta\gamma\\&\qquad+\beta(\gamma-\alpha)\gamma(\alpha-\beta)\gamma\alpha\\&=\alpha\beta\gamma^2[(\alpha-\beta)^2-(\beta-\gamma)(\gamma-\alpha)+(\alpha-\beta)(\beta-\gamma)\\&\qquad+(\alpha-\beta)(\gamma-\alpha)]\\&=\alpha\beta\gamma^2[-(\beta-\gamma)(\gamma-\alpha)];\end{aligned}$$

therefore we have, on dividing down by γ^2,

$$\alpha^2(\gamma-\alpha)^2x^2-2\alpha\beta(\beta-\gamma)(\gamma-\alpha)xy+\beta^2(\gamma-\beta)^2y^2=0;$$

Digitized by Google

therefore $$\alpha(\gamma-\alpha)x = \beta(\beta-\gamma)y;$$

therefore $$\frac{x}{y} = \frac{\beta(\beta-\gamma)}{\alpha(\gamma-\alpha)} = \frac{(b-d)(b-c)}{(a-d)(c-a)} = \frac{(b-d)(c-d)(b-c)}{(a-d)(c-d)(c-a)};$$

therefore, by symmetry,

$$x : y : z = (d-b)(d-c)(b-c) : (d-c)(d-a)(c-a) : (d-a)(d-b)(a-b);$$

therefore $$\frac{x}{(d-b)(d-c)(b-c)} = \frac{y}{(d-c)(d-a)(c-a)} = \frac{z}{(d-a)(d-b)(a-b)};$$

therefore each $$= \frac{x+y+z}{d^2\Sigma(b-c)-d\Sigma(b^2-c^2)+\Sigma bc(b-c)} = \frac{-w}{-(a-b)(b-c)(c-a)};$$

therefore $$x : y : z : w = (d-b)(d-c)(b-c) : (d-c)(d-a)(c-a) : (d-a)(d-b)(a-b) : (b-c)(c-a)(a-b).$$

10342. (Professor NASH, M.A.)—A variable circle passes through a fixed point A on a conic, and meets the conic again in B, C, D; the SIMSON-line of A with respect to the triangle BCD passes through a point whose position is independent of the position and magnitude of the circle. This point lies on the diameter through the image of A with respect to the axis; in the rectangular hyperbola it coincides with the centre.

Solution by R. F. DAVIS, M.A.

Given a fixed parabola, focus F, and a fixed point A outside, from which as focus any variable parabola is described, the circum-circle of the triangle formed by the three common tangents passes through both foci A, F.

Reciprocate with respect to A. Let B, C, D be the three points of intersection of a variable circle passing through A with a fixed conic upon which A lies. Then it is always possible to inscribe a parabola, focus A, to the triangle BCD which will also touch the FRÉGIER line of A with respect to the conic which is the polar reciprocal of F. (See my note in *Reprint*, Vol. LIV., Appendix I.)

Thus the pedal line of A with respect to the triangle BCD passes through a fixed point, the foot of the perpendicular from A on the FRÉGIER line.

15126. (A. M. NESBITT, M.A.) [Suggested by Quest. 9153.]—Prove that the sum of all the integers less than $(p^2+1)^n$ which are divisible by p and not by p^2 is $(p-1)/2p^2\{(p^2+1)^n-1\}^2$, p and n being integral, and that the sum of their squares is $[\{(p-1)X\}/6p^2](2X^2-p^3)$, where

$$X \equiv (p^2+1)^n-1.$$

Digitized by Google

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E., *and the* PROPOSER.

Let $S' =$ sum of numbers $\not> X$ containing p,
$s' =$ number of these numbers.

Then $S' = p + 2p + 3p + \ldots$ ending with X, and $s' = X/p$;
therefore $S' = \frac{1}{2}(X+p)\,X/p$.

Let $S'' =$ sum of numbers $\not> X$ containing p^2,
$s'' =$ number of these numbers.

Then $S'' = p^2 + 2p^2 + 3p^2 + \ldots$ ending with X, and $s'' = X/p^2$;
therefore $S'' = \frac{1}{2}(X+p^2)\,X/p^2$.

Hence the *number sought* $= S' - S'' = \frac{1}{2}X\{(X+p)/p - (X+p^2)/p^2\}$
$= \frac{1}{2}X^2(p-1)/p^2$, as required.

(2) Let $S_2' =$ sum of *squares* of numbers in S'; (their *number* is s')
$= p^2 + 2^2p^2 + 3^2p^2 + \ldots$ ending with X^2; $(s' = X/p)$;
$= p^2 \cdot \frac{1}{6}X/p \cdot (X/p+1)(2X/p+1)$.

Let $S_2'' =$ sum of *squares* of numbers in S''; (their number is s'')
$= p^4 + 2^2p^4 + 3^2p^4 + \ldots$ ending with X^2; $(S'' = X/p^2)$;
$= p^4 \cdot \frac{1}{6}X/p^2 \cdot (X/p^2+1)(2X/p^2+1)$.

Hence the *number sought* $= S_1' - S_2''$
$= \frac{1}{6}X(X+p)\{(2X+p)/p\} - \frac{1}{6}X(X+p^2)(2X+p^2)/p^2$
$= (X/6p^2)(p-1)(2X^2-p^2)$.

5778. (WILKINS.)—If a hyperboloid of one sheet pass through three given lines in space, obtain its centre by a geometrical construction.

Solution by Professor NANSON.

Let a, b, c be the given lines. Let planes through b, c parallel to a intersect in a'; then a' is a generator parallel to a and the line α, mid-way between a, a', passes through the centre. Similarly, we get two other lines Δ, γ through the centre, which is therefore determined.

15158. (H. BATEMAN.)—O is a fixed point on a curve, P a variable point, G is the centre of gravity of the arc OP. Prove that the radius of curvature of the locus of G is $GP^2/s\sin\chi$, where $s =$ arc OP and $\chi =$ angle between GP and the tangent at P.

Solution by R. F. DAVIS, M.A.

Let P′ be a point on the curve near to P. Then, if G′ be the centre of gravity of the arc OP′, (1) GG′ passes through the centre of gravity of the arc PP′, and in the limit when P′ approaches indefinitely near to P the tangent at G to the locus of G passes through P; (2) also $GG' \cdot s = G'P \cdot \delta s$. Angle between the tangents at G and G′ is $\delta s \cdot \sin\chi/G'P$; whence radius of curvature as stated.

PRINTED BY C. F. HODGSON & SON, 2 NEWTON STREET, HIGH HOLBORN, W.C.

Digitized by Google

MATHEMATICAL WORKS

PUBLISHED BY

FRANCIS HODGSON,

89 FARRINGDON STREET, E.C.

In 8vo, cloth, lettered.

PROCEEDINGS of the LONDON MATHEMATICAL SOCIETY.

Vol. I., from January 1865 to November 1866, price 10s.
Vol. II., from November 1866 to November 1869, price 16s.
Vol. III., from November 1869 to November 1871, price 20s.
Vol. IV., from November 1871 to November 1873, price 31s. 6d.
Vol. V., 1873–74, price 15s.
Vol. VI., 1874–75, price 21s.
Vol. VII., 1875–76, price 21s.
Vol. VIII., 1876–77, price 21s.
Vol. IX., 1877–78, price 21s.
Vol. X., 1878–79, price 18s.
Vol. XI., 1879–80, price 12s. 6d.
Vol. XII., 1880–81, price 16s.
Vol. XIII., 1881–82, price 18s.
Vol. XIV., 1882–83, price 20s.
Vol. XV., 1883–84, price 20s.
Vol. XVI., 1884–85, price 20s.
Vol. XVII., 1885–86, price 25s.
Vol. XVIII., 1886–87, price 25s.
Vol. XIX., 1887–88, price 30s.
Vol. XX., 1888–89, price 25s.
Vol. XXI., 1889–90, price 25s.
Vol. XXII., 1890–91, price 27s. 6d.
Vol. XXIII., 1891–92, price 20s.
Vol. XXIV., 1892–93, price 22s.
Vol. XXV., 1893–94, price 22s.
Vol. XXVI., 1894–95, price 30s.
Vol. XXVII., 1895–96, price 35s.
Vol. XXVIII., 1896–97, price 32s. 6d.
Vol. XXIX., 1897–98, 2 parts, price 20s. each.
Vol. XXX., November 1898 to March 1899, price 20s.
Vol. XXXI., April to December, 1899, price 20s.
Vol. XXXII., January to June, 1900, price 20s.
Vol. XXXIII., November 1900 to February 1901, price 20s.
Vol. XXXIV., March 1901 to April 1902, price 20s.

In half-yearly Volumes, demy 8vo, price 6s. 6d. each.
(To Subscribers, price 5s.)

MATHEMATICAL QUESTIONS, with their SOLUTIONS, Reprinted from the EDUCATIONAL TIMES. Edited by D. BIDDLE, M.R.C.S., Fellow of the Royal Statistical Society, formerly Statistician to the General Medical Council.

Of this series seventy-eight volumes (Old Series, 75 Volumes; New Series, 3 Volumes) have now been published, each volume containing, in addition to the papers and solutions that have appeared in the *Educational Times*, about the same quantity of new articles, and comprising contributions, in all branches of Mathematics, from most of the leading Mathematicians in this and other countries.

New Subscribers may have any of these Volumes at Subscription price.

Demy 8vo, price 7s. 6d. Third Edition.

(*Used as the Text-book in the Royal Military Academy, Woolwich.*)

LECTURES on the ELEMENTS of APPLIED MECHANICS. Comprising—(1) Stability of Structures; (2) Strength of Materials. By MORGAN W. CROFTON, F.R.S., late Professor of Mathematics and Mechanics at the Royal Military Academy. Revised by H. HART, M.A.

Third Edition. Extra fcap. 8vo, price 4s. 6d.

ELEMENTARY MANUAL of COORDINATE GEOMETRY and CONIC SECTIONS. By Rev. J. WHITE, M.A.

Digitized by Google

Thirteenth Edition, small crown 8vo, price 2s. 6d.

AN INTRODUCTORY COURSE OF PLANE TRIGONOMETRY AND LOGARITHMS.

By JOHN WALMSLEY, B.A.

"This book is carefully done; has full extent of matter, and good store of examples."—*Athenæum.*

"This is a carefully worked out treatise, with a very large collection of well-chosen and well-arranged examples."—*Papers for the Schoolmaster.*

"This is an excellent work. The proofs of the several propositions are distinct, the explanations clear and concise, and the general plan and arrangement accurate and methodical."—*The Museum and English Journal of Education.*

"The explanations of logarithms are remarkably full and clear. . . . The several parts of the subject are, throughout the work, treated according to the most recent and approved methods. . . . It is, in fact, a book for *beginners*, and by far the simplest and most satisfactory work of the kind we have met with."—*Educational Times.*

A KEY

to the above, containing Solutions of all the Examples therein, price Five Shillings (net).

By the same Author.

New Edition, fcap. 8vo, cloth, price 5s.

PLANE TRIGONOMETRY AND LOGARITHMS.

FOR SCHOOLS AND COLLEGES. Comprising the higher branches of the subject not treated in the elementary work.

"This is an expansion of Mr. Walmsley's 'Introductory Course of Plane Trigonometry,' which has been already noticed with commendation in our columns, but so greatly extended as to justify its being regarded as a new work. . . . It was natural that teachers, who had found the elementary parts well done, should have desired a completed treatise on the same lines, and Mr. Walmsley has now put the finishing touches to his conception of how Trigonometry should be taught. There is no perfunctory work manifest in this later growth, and some of the chapters—notably those on the imaginary expression $\sqrt{-1}$, and general proofs of the fundamental formulæ—are especially good. These last deal with a portion of the recent literature connected with the proofs for $\sin(A+B)$, &c., and are supplemented by one or two generalized proofs by Mr. Walmsley himself. We need only further say that the new chapters are quite up to the level of the previous work, and not only evidence great love for the subject, but considerable power in assimilating what has been done, and in representing the results to his readers."—*Educational Times.*

IN PREPARATION.

By the same Author.

Suitable for Students preparing for the University Local and similar Examinations.

AN INTRODUCTION TO MECHANICS.

With numerous Examples, many of which are fully worked out in illustration of the text.

Seventh Edition, crown 8vo.

MODERN SIDE ARITHMETIC: EXAMPLES ONLY.

By the Rev. T. MITCHESON, B.A., Assistant-Master in the City of London School.

Part I., Crown 8vo, cloth, pp. 119, 1s. (Answers, 1s; Teachers' Copy, with Answers, 1s. 6d.)

Part II., Crown 8vo, cloth, pp. 200, 1s. 6d. (Answers, 1s.; Teachers' Copy, with Answers, 2s.)

Complete in One Volume, crown 8vo, cloth, 2s. (Answers, 1s. 6d.; Teachers' Copy, with Answers, 3s.)

By the same Author. New and enlarged Edition.

EXAMPLES IN ALGEBRA.

Crown 8vo, cloth. Pp. 96, 1s.; with Answers, 1s. 6d. Answers only, 1s.

Digitized by Google

Now ready, Royal 8vo, 967 pp., with 285 Diagrams, price 34s.

A SYNOPSIS

OF ELEMENTARY RESULTS IN

PURE MATHEMATICS,

Containing Six Thousand Propositions, Formulæ, and methods of Analysis, with abridged Demonstrations; supplemented by an Index to the articles on Pure Mathematics which have appeared in thirty-two BRITISH AND FOREIGN JOURNALS AND TRANSACTIONS *during the present century, containing about* 14,000 *references.*

BY G. S. CARR, M.A.,

Late Prizeman and Scholar of Gonville and Caius College, Cambridge.

The work may also be had in Sections separately, as follows:—

		s.	d.
Section	I.—MATHEMATICAL TABLES; including C. G. S. units. Least Factors from 1 to 99,000. The Gamma-function, &c. ..... Price	2	0
„	II.—ALGEBRA	2	6
„	III.—THEORY OF EQUATIONS AND DETERMINANTS	2	0
„	IV.—PLANE TRIGONOMETRY { together; with		
„	V.—SPHERICAL do. { 22 Diagrams. }	2	0
„	VI.—ELEMENTARY GEOMETRY; with 47 Diagrams	2	6
„	VII.—GEOMETRICAL CONICS; with 35 Diagrams	2	0
„	VIII.—DIFFERENTIAL CALCULUS	2	0
„	IX.—INTEGRAL CALCULUS	3	6
„	X.—CALCULUS OF VARIATIONS		
„	XI.—DIFFERENTIAL EQUATIONS	3	6
„	XII.—CALCULUS OF FINITE DIFFERENCES		
„	XIII.—PLANE COORDINATE GEOMETRY, with 182 Diagrams	6	0
„	XIV.—SOLID COORDINATE GEOMETRY, with 26 Diagrams	3	6

N.B.—The Index will not be sold separately, but possessors of previous sections may complete the volume by returning these to the publisher to be bound up with the remaining sections and the Index.

The work is in part designed for the use of University and other Candidates who may be reading for examination. It forms a digest of the contents of ordinary treatises and is arranged so as to enable the student rapidly to revise his subjects. To this end all the important propositions in each branch of Mathematics are presented within the compass of a few pages. This has been accomplished, firstly, by the omission of all extraneous matter and redundant explanations, and, secondly, by carefully compressing the demonstrations in such a manner as to place only the leading steps of each prominently before the reader. The whole is intended to form a permanent work of reference for mathematicians.

NOTICES OF THE PRESS.

"The completed work admirably serves the object Mr. Carr set before himself. Every subject that can be classed under the head of Pure Mathematics, with the exception perhaps of Quaternions, appears to us to have been carefully treated on the author's lines."—*The London, Edinburgh, and Dublin Philosophical Magazine.*

"We believe the book to be singularly free from errors. The author is to be heartily congratulated on the successful termination of his labours. The work is printed in a good bold type on good paper, and the figures are admirably drawn."—*Nature.*

"By the boldly printed reference numbers the reader can, if he pleases, go back and back upon the links in the chain of demonstration from the highest theorem to elementary principles. Repetition, otherwise necessary, is thereby avoided, and at the same time the connection of theorems and the course of evolution of formulæ are more clearly exhibited. Although the object of the synopsis is to supplement only, and not to supersede, the usual text-books, the clearness of statement is such that, without any other book, a student who has already some knowledge of mathematics will find it a great saving of time to use this volume as a guide to higher attainments. We cannot adequately set forth in this notice the high estimation we have of this work. The engravings in it are numerous and well executed, especially those referring to geometry and geometrical conics, where a distinguishing thick line is employed with excellent effect. The work is beautifully printed."—*Engineering.*

"The idea of the work is well conceived and is well carried out. In its completed form it is as full, and yet as concise, an index as could be desired of all the most important propositions in the extensive field of Pure Mathematics."—*The Academy.*

Digitized by Google

New and Revised Edition, small crown 8vo, price 2s.

AN INTRODUCTION TO GEOMETRY

FOR THE USE OF BEGINNERS.

CONSISTING OF

EUCLID'S ELEMENTS, BOOK I.

ACCOMPANIED BY NUMEROUS EXPLANATIONS, QUESTIONS, AND EXERCISES.

BY JOHN WALMSLEY, B.A.

This work is characterized by its abundant materials suitable for the training of pupils in the performance of *original work*. These materials are so graduated and arranged as to be specially suited for class-work. They furnish a copious store of useful examples from which the teacher may readily draw more or less, according to the special needs of his class, and so as to help his own method of instruction.

OPINIONS OF THE PRESS.

"We cordially recommend this book. The plan adopted is founded upon a proper appreciation of the soundest principles of teaching. We have not space to give it in detail, but Mr. Walmsley is fully justified in saying that it provides 'for a natural and continuous training to pupils taken in classes.'"—*Athenæum.*

"The book has been carefully written, and will be cordially welcomed by all those who are interested in the best methods of teaching Geometry."—*School Guardian.*

"Mr. Walmsley has made an addition of a novel kind to the many recent works intended to simplify the teaching of the elements of Geometry. . . . The system will undoubtedly help the pupil to a thorough comprehension of his subject."—*School Board Chronicle.*

"When we consider how many teachers of Euclid teach it without intelligence, and then lay the blame on the stupidity of the pupils, we could wish that every young teacher of Euclid, however high he may have been among the Wranglers, would take the trouble to read Mr. Walmsley's book through before he begins to teach the First Book to young boys."—*Journal of Education.*

"We have used the book to the manifest pleasure and interest, as well as progress, of our own students in mathematics ever since it was published, and we have the greatest pleasure in recommending its use to other teachers. The *Questions* and *Exercises* are of incalculable value to the teacher."—*Educational Chronicle.*

KEY to the above, price 3s.

Crown 8vo, price 1s. 6d.

THE ELEMENTS OF LOGARITHMS; with the Sandhurst and Woolwich Questions for 1880-87. By W. GALLATLY, M.A.

Demy 8vo, price 5s.

ALGEBRA IDENTIFIED WITH GEOMETRY, in Five Tracts. By ALEXANDER J. ELLIS, F.R.S., F.S.A.

Demy 8vo, price 5s. each.

TRACTS relating to the MODERN HIGHER MATHEMATICS. By the Rev. W. J. WRIGHT, M.A.

TRACT No. 1.—DETERMINANTS. TRACT No. 2.—TRILINEAR COORDINATES. TRACT No. 3.—INVARIANTS.

The object of this series is to afford to the young student an easy introduction to the study of the higher branches of modern Mathematics. It is proposed to follow the above with Tracts on Theory of Surfaces, Elliptic Integrals and Quaternions.

Digitized by Google

Digitized by Google

Digitized by Google

Digitized by Google

3 2044 102 915 691

Digitized by Google